THE PHOTO-DISINTEGRATION OF THE DEUTERON

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Chia Hsien Teng

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F. J. Belinfante PROFESSOR IN CHARGE OF THESIS

K. Lark-Horovitz / Whaley HEAD OF SCHOOL OR DEPARTMENT

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ABSTRACT

We have calculated the cross-section for photo-disintegration of the deuteron by a high energy photon. The interaction between the photon and the nucleons was expressed as

$$W = -\int \vec{I}\cdot\vec{A}\, d^3\vec{x} = -\int \rho\, \vec{\xi}\, \frac{\vec{E}(\vec{\xi}) + \vec{E}(0)}{2}\, d^3\vec{x} - \int \vec{M}\cdot \frac{2\vec{H}(\vec{\xi}) + \vec{H}(0)}{2}\, d^3\vec{x} + O + \frac{d}{dt}\cdots \quad \text{(a)}$$

where

$$O = +\sum_{ijk} \int \rho\, \xi_i \xi_j \xi_k \left(\frac{1}{4} \nabla_i \nabla_j E_k - \frac{1}{6} \nabla_i \nabla_k E_j\right) d^3\vec{x} + \frac{1}{6} \sum_{ijk\ell} \int \xi_i \xi_j \xi_k I_\ell (\nabla_i \nabla_k \nabla_\ell A_j)\, d^3\vec{x} + \cdots$$

stands for further electric and magnetic octupole and higher multipole interactions as far as not included already with the dipole and quadrupole interactions in the first two terms of the right hand side of eq.(a) Compare eq.(39a)). ρ and $\vec{I}$ are total charge and current density, but for explicit calculations the meson charge ("exchange charge") was neglected and

$$\rho = e\, \Psi^\dagger \frac{1+\tau_z}{2} \Psi \qquad (13)$$

was used. This approximation, satisfactory at energies below 50 Mev is not quite so good at the energies > 90 Mev that were considered in this thesis.

$\vec{\xi}$ are the coordinates measured from the center of gravity of the deuteron. $\vec{E}(\vec{\xi})$ and $\vec{H}(\vec{\xi})$ stand for the electric and magnetic field at distance $\vec{\xi}$ from this center of gravity, $\vec{A}$ for the vector point at this point, $d^3\vec{x} = d^3\vec{\xi}$. $\vec{M}$ is the magnetization of the deuteron at this point; it is defined by

$$\vec{M} = \frac{1}{2} \vec{\xi} \times \vec{I} \qquad (35)$$

where $\vec{I}$ will include meson currents. As we have insufficient information on the properties of meson, eq.(35) was for practical calculation replaced by (compare eq.(9))

$$\vec{M} = \Psi^\dagger \frac{e\hbar}{2Mc} \left(\frac{1+\tau_z}{2} \mu_p + \frac{1-\tau_z}{2} \mu_N \right) \vec{\sigma}\, \Psi, \qquad (6)$$

the integral of which is experimentally known to be a good approximation for the magnetic dipole moment of the deuteron. As only central forces were taken into account for the interaction of the nucleons through the meson field, this approximation would be sufficient. However there are no reasons for believing that the magnetic quadrupole interactions calculated for (b) should be correct.

The term $\frac{d}{dt}\ldots$ in (a) can be omitted as only first order perturbation theory was used in calculating the cross-sections. The central forces between the nucleons were taken as a 50-50 mixture of Majorana forces and of ordinary forces as proposed by Christian and Hart (compare eq.(41d)

$$U = \frac{1}{2}\left\{1 + (-1)^{\ell}\right\} V(r) \qquad (c)$$

There is little reason to believe that this potential is correct, and for the ground state of the deuteron, where it was also used, its lack of spin dependence is definitely in disagreement (as eq.(c) would give the same binding for a ^1S state as for the ^3S state). However no better interaction potential for the high energy region is known today. For $V(r)$ a square well of depth V_0 and effective range r_0 was taken (see eq.(49)).

The cross-section may be divided into an electric cross-section σ^E for transition with $\Delta m_s = 0$, if the photon enters along the z-axis, and a magnetic cross-section σ^M for transitions with $\Delta m_s = \pm 1$ in this case. The expansion in multipole contributions on which the calculations are based, do not converge quite so good at energies >90 Mev for the photon. This is the reason why different approximations made for transitions to final states with $\ell = 0$, or $\ell = 1$ gave substantially different numerical results. The formulas obtained for the cross-sections are:

σ^E for ~~the~~ final state with $\ell = 1$:

$$\sigma_{\ell=1} = \frac{\pi}{6}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\, k \left| \sqrt{\mathcal{L}} \int_0^\infty rF_{f1}(r)\, \frac{1+\sqrt{\frac{\pi}{kr}}J_{1/2}(k\frac{r}{2})}{2} F_i(r)\,dr \right|^2 \quad (22)$$

$$\sigma_d = \frac{\pi}{6}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\, k \left| \sqrt{\mathcal{L}} \int_0^\infty rF_{f1}(r)\, F_i(r)\,dr \right|^2 \quad (23)$$

σ^E for final state with $\ell = 2$:

$$\sigma_{\ell=2} = \frac{\pi}{120}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\, k \left| \sqrt{\mathcal{L}} \int_0^\infty rF_{f2}(r)\; 3\sqrt{\frac{\pi}{kr}}\, J_{\frac{3}{2}}(k\tfrac{r}{2})\, F_i(r)\,dr \right|^2 \quad (26)$$

$$\sigma_q = \frac{\pi}{120}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\, k \left| \sqrt{\mathcal{L}} \int_0^\infty rF_{f2}(r)\; k\frac{r}{2}\, F_i(r)\,dr \right|^2 \quad (28)$$

σ^M for final state with $\ell = 0$:

$\sigma_{md} = 0$ (compare eq.(10))

for final state with $\ell = 1$ (hoping the best for the choice eq.(b)for M):

$$\sigma_{mq} = \frac{1}{9}\left(\frac{E_\gamma}{Mc^2}\right)^2 \left(\mu_p^2 + \mu_N^2 - \frac{2}{3}\mu_p\mu_N\right)\sigma_d \quad (29)$$

Here E_γ = $\hbar ck$ = the incident photon energy in the center of gravity system.

E_f = the final energy of the nucleons in the center of gravity system.

M = nucleon mass.

$\mathcal{L}$ = radius of spherical volume $V = \frac{4\pi}{3}\mathcal{L}^3$, in which we normalized the final spatial wave function of the disintegrated deuteron,

$$u_{f\ell}(\vec{r}) = \frac{F_{f\ell}(r)}{r}\, Y_{\ell,m}(\theta,\varphi); \quad (46)$$

while the spatial wave function of the deuteron in its ground state was

$$u_i(\vec{r}) = \frac{1}{(4\pi)^{1/2}}\frac{F_i(r)}{r}\,.$$

$\sigma_{\ell=1}$ and $\sigma_{\ell=2}$ were **found** expanding the expression $e^{i\vec{k}\cdot\vec{\xi}} = e^{ik\xi\cos\theta}$ appearing in the matrix element by

$$\sqrt{\frac{\pi}{2k\xi}}\sum_{\ell'=0}^{\infty} i^{\ell'}(2\ell'+1)J_{\ell'+\frac{1}{2}}(k\xi)\, P_{\ell'}(\cos\theta)$$

and neglecting terms with $l' \geq 2$. σ_d, σ_q, σ_{md}, σ_{mq} were calculated using a Taylor-expansion for $e^{i\vec{k}\vec{\xi}}$ and using only the first two terms $(1 + i k \xi \cos\theta)$. Numerical results can be found below eq.(28) and in section 3. Our eq. (29) is different from the formula published by Marshall and Guth. All cross-sections were calculated in the center of gravity system of photon and deuteron. The energy of the photon in the laboratory system is

$$E_\gamma' = E_\gamma \sqrt{\frac{\sqrt{4M^2c^4 + E_\gamma^2} + E_\gamma}{\sqrt{4M^2c^4 + E_\gamma^2} - E_\gamma}} ,$$

which will be about 9% more than E_γ for $E_\gamma \sim 150$ Mev.

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THE PHOTO-DISINTEGRATION OF THE DEUTERON

1. INTRODUCTION

Since the high energy machine has been well developed, high energy photo-disintegration of nuclei is more interesting than ever before. But the present knowledge of the neutron-proton interaction at high energies is not quite clear. From low energy neutron-proton scattering one can obtain the effective range of the neutron-proton interaction[1,2], but little information about the shape and exchange character of the potential. High energy neutron-proton scattering should yield information concerning the shape of the potential and might give a check on its exchange character. If we assume that there are only four types of interactions, i.e. the ordinary force, Majorana force, Bartlet force and Heisenberg force, we need four parameters to characterize the exchange character of the potential. This presumes that the particles interact by central forces only that do not depend on velocity, which is doubtful at very high energies. Theoretical interpretation of the data available at present by Christian and Hart[3] indicates that if one arbitrarily neglects Bartlet and Heisenberg forces, then the exchange character of the potential seems best represented by a mixture of 50 percent of ordinary force and 50 percent of Majorana force, or the percentage of Majorana force seems to be somewhat higher than 50 percent, but the experimental evidence is not complete enough to assign a definite value. So Christian and Hart only determined two parameters. Since the experimental results are so inaccurate, we cannot hope for more parameters than these determined by them. From the theoretical point of view, these results are quite arbitrary, because there is no sound theoretical basis why the exchange character of the nuclear potential should be a mixture of ordinary force and Majorana force only. In the following we want to derive what Christian and Hart's interaction would predict for high energy photo-disintegration of the deuteron. If this would definitely not agree with experimental results on photodisintegration, we would have evidence that Christian and Hart's assumption has been all too simple.

It has been shown by Møller and Rosenfeld[4] that the contributions of virtual mesons to the electric dipole moment and electric quadrupole moment of the two-nucleon system are zero to order $v_{nucleon}/c$. This makes it possible to calculate the cross-section for photo-electric disintegration of the deuteron at low energies entirely in terms of the interaction potentials between neutron and proton in triplet states of various orbital angular momenta. Guth and Marshall[5] pointed out that the contributions of the virtual mesons for energies below 150 Mev can be neglected. (They believe that multipoles other than dipole and quadrupole begin to become important for energies higher than 150 Mev.) For the photo-magnetic effect, the situation is different. It may be true that the magnetic exchange moment (that is, the magnetic moment due to the currents of virtual mesons in the deuteron,) vanishes for any stationary state of the deuteron, although the exchange moments of H^3 and He^3 amount to about six percent.[6] Pais[7] pointed out that the exchange currents are likely to contribute to the photomagnetic cross-section. Although his original estimate of the magnitude of this contribution on the basis of Møller and Rosenfeld's mixed theory was much too large[8], the exact magnitude of this contribution is still unknown. The same holds for magnetic quadrupole transitions.

In the present paper the cross-sections for the electric dipole and quadrupole transition and magnetic quadrupole transitions have been calculated by using a square well potential for two different effective ranges: 1.56×10^{-13}cm. and 1.74×10^{-13}cm. We obtain these results following

Guth and Marshall's point of view, but also used somewhat different expressions for the cross-sections. These differences are due to different ways of expanding the factor exp $(i\vec{k}.\vec{x})$ representing the photon wave in the matrix element. The fact that this has an appreciable influence on the cross-sections for the electric "dipole" transition shows that the expansion in multipole contributions used by Guth and Marshall cannot really be assumed to converge rapidly at energies around 100 Mev as assumed by these authors. Furthermore, we think that there is no sound theoretical basis to ignore the contributions of the virtual mesons at these energies. In the next section, we shall give a detailed discussion of these points. Section 3 contains all the results we obtained, and the graphs which show the variation of the cross-sections for different energies and for two different effective ranges of the square well potentials. Section 4 presents the mathematical appendix which gives the derivations of all the important formulas appearing in the text.

All calculations were made by taking 2.237 Mev as the binding energy of the deuteron, and a 50-50 mixture of ordinary and Majorana forces is assumed for both triplet and singlet states of the deuteron.

2. DETAILED DISCUSSION

The photodisintegration of the deuteron can be described as a transition of a two-nucleon system of charge e, under absorption of a photon, from a bound state (the deuteron in its ground state) to a state in the continuum of the energy spectrum ("disintegrated deuteron" state). The probability per second for this transition, for a system with one deuteron and one photon initially within a volume V, is calculated by means of the well-known equation

$$w = \frac{2\pi}{\hbar} \rho(E) \left| W \right|^2 \qquad (1)$$

where W is the coupling between nucleons and photons, treated as a small perturbation. If more properly we consider the deuteron as a system consisting of two nucleons and a fluctuating number of virtual mesons, then W should be the coupling of nucleons and mesons with photons. This coupling may be written as

$$W = -\int \vec{I} \cdot \vec{A} \; d^3\vec{x} \qquad (2)$$

where $\vec{I}$ is the sum of the nucleon current $\vec{I}_n$, the meson current $\vec{I}_m$, and the terms $\vec{I}_{eg}$ which are products of a nucleon wave function, a conjugate nucleon wave function, a meson wave function, and constants, and $\vec{A}$ is the transverse photon field. If e denotes the elementary charge, and g the meson-nucleon coupling constant, the $\vec{I}_{eg}$ contains constants e x g. If in $\vec{I}_m$ and $\vec{I}_{eg}$ we substitute for the meson fields, the solutions of the static equations of the meson field given by Møller and Rosenfeld, then $\vec{I}_m + \vec{I}_{eg}$ represents what is often called the "exchange current" of the nucleonic system.

In W we may expand the photon field $\vec{A}(x)$ in transverse plane waves proportional to $e^{i\vec{k}\cdot\vec{x}}$ where $\vec{k}$ is the propagation vector of the photon field.

The current $\vec{I}$, in particular, its matrix element corresponding to a transition of the nucleons from the bound state to the disintegrated state, will be negligible for distances r between the nucleons which are larger than the size of the deuteron (the range of the wave function of the deuteron 4.3×10^{-13}cm. which is considerably more than the effective range of nuclear forces, which is only around 1.6×10^{-13}cm.). If $\vec{R}$ is the radius vector of the center of gravity of the deuteron and $\vec{r}$ the relative coordinate of the two nucleons, and we use $\vec{R}$ and $\vec{r}$ instead of the positions $\vec{x}_1$ and $\vec{x}_2$ of the two nucleons as our spatial variables, then $\vec{I}$ comes multiplied in W by factors $e^{i\vec{k}.\vec{R}}$ $e^{i\vec{k}.\frac{\vec{r}}{2}}$. The last factor may be expanded in powers of $(\vec{k}. \frac{\vec{r}}{2})$, which corresponds to an expansion of W in multipole contributions. We may also expand $e^{i\vec{k}.\frac{\vec{r}}{2}}$ in terms of Bessel functions of half integral order of argument $(k\frac{r}{2})$ and Legendre polynomials in cos θ, if θ is the angle between $\vec{k}$ and $\vec{r}$. Such expansion will converge rapidly if $e^{i\vec{k}.\frac{\vec{r}}{2}}$ is used only in a region where $(\vec{k} \cdot \frac{\vec{r}}{2})$ remains small, that is, if the size of the deuteron of 4.3×10^{-13}cm. is small compared with the wave length $= 2\pi/k$ of the photon. We find

Energy of photon in Mev.	Wave length of photon in $\times 10^{-12}$cm.	Value of kr/2 where r is deuteron size (for $\theta = 0$)
9	13.5	0.10
23	5.4	0.25
46	2.7	0.50
92	1.35	1.00

From this it is obvious that $e^{i\vec{k}.\frac{\vec{r}}{2}}$ can be considered approximately equal to 1 $(|e^{ik.r/2} - 1| \lesssim \frac{1}{4}$, $\vec{k} \cdot \vec{r}/2 \lesssim \frac{1}{4}$) only for photons of energy lower than 23 Mev; that a linear expansion of $e^{i\vec{k}.\frac{\vec{r}}{2}} (= 1 + i\vec{k}.\frac{\vec{r}}{2}$) is a reasonable approximation only (for $k\,r/2 < 0.50$, that is) for photons of energy lower than 46 Mev; that (for $k\,r/2 > 1$, that is) for photons of energy more than 92 Mev even a quadratic expansion $(1 + i\vec{k}\cdot\frac{\vec{r}}{2} - \frac{1}{2}(\vec{k}\cdot\frac{\vec{r}}{2})^2)$ would be insufficient. In fact the expansions are a little better than the above would make us believe that, as for values of $\theta \neq 0$, $k\,r/2$ will appear multiplied by $\cos\theta$ and the energy of the photon may be larger by a factor $\sec\theta$ for obtaining the same value of $(\vec{k}\cdot\vec{r}/2)$. Let us therefore say that the limits of validity for our expansions of $e^{i\vec{k}\cdot\vec{r}/2}$ are about twice as high as calculated here. Yet, as electric and magnetic dipole transitions and electric quadrupole transitions correspond only to the terms $1 + i\vec{k}\cdot\frac{\vec{r}}{2}$ with above expansion, we still cannot trust this to be sufficient approximation at energies above $2 \times 46 = 92$ Mev, that is, in the energy region considered in the present paper. The magnetic quadrupole contribution cannot be calculated without knowing the distribution of the magnetization of the deuteron over space. As we do not trust the magnetic moment as calculated by meson theory and on the other hand we do not have experiments at our disposal giving the magnetic quadrupole moment of the deuteron, we have to content ourselves at present in the high energy region > 90 Mev with an insufficient approximation. Nevertheless, we calculated the cross-section for magnetic quadrupole transition by following Guth and Marshall's point of view merely for the sake of comparison with their results.

As we consider only transition with conservation of energy, we may replace W by $W + \frac{dX}{dt}$, where X is any quantity (since

$$\frac{dX}{dt} = \frac{i}{\hbar}(HX - XH)$$

and H has the same value before and after the transition). We may use this possibility for changing W into

$$W = -\int \rho\vec{\xi}\cdot E(\vec{\xi})\, d^3\vec{x} - \int \vec{M}\cdot\vec{H}\, d^3\vec{x} + \frac{1}{2}\sum_i\sum_j \int \rho\,\xi_i\,\xi_j\,\nabla_i E_j\, d^3\vec{x} - \frac{1}{2}\sum_i\sum_j\sum_k \int [illegible]\, d\vec{x} \quad (3)^*$$

* See section 4 for eq.(3) and for other important formulas appearing in the following.

where $\vec{\xi}$ is the distance from the center of gravity of the nucleons to the point $\vec{x}$ over which we integrate, and i, j, k run from 1 to 3, standing for x, y, z components of the vectors, and the nabla ∇ stands for partial differentiation with respect to x, y, or z. The last term is obviously of the same dimension as the magnetic quadrupole contribution, and will be left out of consideration, because we do not know anything definite about the magnetic quadrupole moment. By Taylor's expansion, the third term can be approximated by

$$\frac{1}{2}\sum_i \int \rho \xi_i \left[E_i(\vec{\xi}) - E_i(0) \right] d^3\vec{x}, \tag{4}$$

if we neglect the higher order terms in the expansion (electric octupole terms and so on, which may be of the order of magnitude of the magnetic quadrupole or perhaps even less). In the above expression $\vec{E}(0)$ and $\vec{E}(\vec{\xi})$ stand for the electric field evaluated at the center of gravity of the nucleons and the point $\vec{x}$ respectively. The second term $-\int \vec{M}\cdot\vec{H}\, d^3\vec{x}$ stands for the magnetic effect as far as this is taken into account, and is of the form

$$-\int \vec{M}\cdot\vec{H}\, d^3\vec{x} = -\frac{1}{2}\int \vec{H}\cdot(\vec{\xi}\times\vec{I})\, d^3\vec{x}. \tag{5}$$

So finally, eq. (3) becomes

$$W = -\frac{1}{2}\int \rho\vec{\xi}\cdot\vec{E}(\vec{\xi})d^3\vec{x} - \frac{1}{2}\int \rho\vec{\xi}\cdot\vec{E}(0)d^3\vec{x} - \int \vec{M}\cdot\vec{H}\, d^3\vec{x} \tag{6}$$

This expression includes the electric quadrupole contributions, as we will show below.

We can consider the following transitions of the nucleonic system.

(I) $(\ell = 0) \longrightarrow (\ell = 0)$

(II) $(\ell = 0) \longrightarrow (\ell = 1)$

(III) $(\ell = 0) \longrightarrow (\ell = 2)$

(IV) $(\ell = 0) \longrightarrow (\ell \geq 3)$.

Here ℓ stands for the quantum number for the angular momentum of the final state of the two nucleons. Transition (I) is possible by eq.(1) through the last term in eq. (6) only. Transition (II) is possible by eq. (1) through the first two terms in eq. (6). Transition (III) is possible through the first term in eq. (6). Transitions (II) and (III) are also possible through the last term of eq. (6), but since we do not know M as a function of $\vec{\xi}$, but we only know $\int \vec{M}\, d^3\vec{x}$ from experiments on magnetic moments, these contributions cannot be calculated (For the same reason we omitted the last term of eq. (3)). Transition (IV) are completely neglected for the same reasons as given for the approximation made in eq. (4), although for our energy range the contributions of higher order multipoles may be not quite unimportant. Note that there are never simultaneous contributions for the magnetic as well as the electric terms in eq. (6) to one and the same transition of the nucleonic system, because of the selection rules.

For $\int \vec{M}\, d^3\vec{x}$ we may write

$$\int \Psi^\dagger \frac{e\hbar}{2Mc} \left(\frac{1+\tau_z}{2} \mu_p + \frac{1-\tau_z}{2} \mu_N \right) \vec{\sigma}\, \Psi\, d^3\vec{x}\,, \tag{7}$$

where Ψ is the quantized nucleonic wave function, M is the mass of the proton, and τ_z is the "isotopic spin" operator, of which the eigenvalue is taken to be +1 for proton, and -1 for neutron, while μ_p and μ_N are the experimental magnetic moments of proton and neutron respectively. The matrix elements of (7) are given by

$$\int \Psi_f^* \left[\sum_{j=1}^{2} \frac{e\hbar}{2Mc} \left(\frac{1+\tau_{zj}}{2} \mu_p + \frac{1-\tau_{zj}}{2} \mu_N \right) \vec{\sigma}_j \right] \Psi_i\, d^3\vec{x}_1\, d^3\vec{x}_2 \tag{8}$$

where Ψ_i and Ψ_f are the initial and final two-particle wave functions of the nucleonic system, and $\vec{x}_1$ and $\vec{x}_2$ are the position coordinates of the two nucleons. If tentatively we take

$$M_{fi} = \Psi_f^* \sum_{j=1}^{2} \frac{e\hbar}{2Mc} \left(\frac{1+\tau_{zj}}{2} \mu_p + \frac{1-\tau_{zj}}{2} \mu_N \right) \Psi_i\,, \tag{9}$$

(which may easily be wrong by the divergence of a dyadic $\sum_{k=1}^{3} \nabla_k T^{k\ell}$ that by integration over space would not contribute to the total magnetic moment), then the average square of the matrix element for the transition (I) is

$$\left|\langle \ell = 0 \,|\, W^M \,|\, \ell = 0\rangle\right|^2_{Av} = \left|\langle \ell=0 \,|\, -\int \vec{M}\cdot\vec{H}\, d^3\vec{x} \,|\, \ell=0\rangle\right|^2_{Av}$$

$$= \frac{2\pi c^2\hbar}{V\omega}\left(\frac{e\hbar}{2Mc}\right)^2 k^2 \left[\mu_P^2 + \mu_N^2 + \frac{2}{3}\mu_P\mu_N\right] \left|\int u^*_{f0}(\vec{r})\, u_i(\vec{r})\, d^3\vec{r}\right|^2 \qquad (10)$$

Here $\vec{r}$ is the relative coordinate of the two nucleons, $\omega = ck$ is the "angular frequency" of the photon wave, and $\hbar ck = E_\gamma$ is the energy of the photon (incident γ-ray) in the center-of-mass system, u_i is the spatial wave function for the ground state, and $u_{f\ell}$ is that for the state of angular momentum ℓ, while V is the volume in which the photon wave and the nucleonic system are enclosed. Furthermore, the symbol Av means the average over the two directions of circular polarizations and over the substates of the spin of the initial state and sum over the substates of spin and isotopic spin of the final state. Our result (10) vanishes, since $u_i(\vec{r})$ and $u_{f0}(\vec{r})$, which are two wave functions corresponding to two energies of the nucleonic system differing by the energy of the absorbed photon, are orthogonal to each other.

For the calculation of the matrix elements of the first term in eq.(6) for the transitions (II) and (III), we expand the electric field of the incident photon

$$\vec{E} = \frac{1}{c}\sum_k \sum_{P=L,R} \sqrt{\frac{2\pi c^2\hbar}{V\omega}}\,(i\omega)\, \vec{e}^{\,P}_k \left(a_{kP}\, e^{+i\vec{k}\cdot\vec{x}} - a^\dagger_{kP}\, e^{+i\vec{k}\cdot\vec{x}}\right) \qquad (11)$$

Here, a_{kP} and $a^\dagger_{kP}$ are the annihilation and creation operators for the photon, and the values L and R of the label P denote left and right hand polarization, according to

$$\vec{e}^{\,L}_k = \frac{1}{\sqrt{2}}\left(\vec{e}^{\,1}_k + i\,\vec{e}^{\,2}_k\right) \qquad (12a)$$

$$\vec{e}^{\,R}_k = \frac{1}{\sqrt{2}}\left(\vec{e}^{\,1}_k - i\,\vec{e}^{\,2}_k\right) \qquad (12b)$$

where $\vec{e}^{\,1}_k$, $\vec{e}^{\,2}_k$ and $\vec{e}^{\,3}_k = \vec{k}/k$ are three unit vectors along the x, y, and z axes.

Further we take for the matrix element of ρ :

$$\rho_{fi} = e\, \Psi_f^{*} \Big(\sum_{j=1}^{2} \frac{1+\tau_{3j}}{2} \Big) \Psi_i \tag{13}$$

where we neglected the large exchange charge density, because of lack of decisive experimental information of the type of meson interaction between the nucleons. Introducing $\vec{R}$ as the coordinate of the center of gravity of the deuteron, and $\vec{r}$ as the relative coordinate of the two nucleons ($\vec{r} = \vec{x}_2 - \vec{x}_1$) as we did before, We find the average square of the matrix element of the first two terms in eq. (6)

$$|\langle f|W^E|i\rangle|^2_{Av} = \frac{2\pi\hbar\omega}{V}\, e^2 \left| \frac{1}{2} \int u_{f\ell}^{*}(\vec{r})\ \vec{e}_k^{\,P} \cdot \frac{\vec{r}}{2}\ \big(e^{i\vec{k}\cdot\frac{\vec{r}}{2}} + 1 \big)\, u_i(\vec{r})\, d^3\vec{r} \right|^2_{Av}. \tag{14}$$

Here the symbol Av has the same meaning as before. Remembering that $\vec{k} = k\, \vec{e}_k^{\,3}$ is taken along the z - direction, we can expand $e^{i\vec{k}\cdot\vec{r}/2}$ by the well-known expansion

$$e^{i\vec{k}\cdot\frac{\vec{r}}{2}} = e^{ik\frac{r}{2}\cos\theta} = \sqrt{\frac{\pi}{2k\frac{r}{2}}} \sum_{\ell'=0}^{\infty} i^{\ell'} (2\ell'+1)\, J_{\ell'+\frac{1}{2}}(k\tfrac{r}{2})\, P_{\ell'}(\cos\theta) \tag{15}$$

For a given value of ℓ, we need here only consider terms with $\ell' = \ell \pm 1$, because, for a given ℓ, $|\langle f|W^E|i\rangle|^2_{Av}$ vanishes for ℓ' other than $\ell' = \ell \pm 1$. In fact we need only consider $\ell' = 0$ for transition (I) and $\ell' = 1$ for transition (II). The terms with $\ell' \geq 2$ would correspond to higher order multipoles. In order to compare the expansion (15) with the expansion in multipoles by Taylor's expansion of $e^{i\vec{k}\cdot\vec{r}/2}$, which would give the contributions of different multipoles to the cross-section from Møller and Rosenfeld's or Guth and Marshall's point of view, we write out the explicit forms of Bessel functions of first two half-integral orders

$$J_{\frac{1}{2}}(k\tfrac{r}{2}) = \sqrt{\frac{2}{\pi k\frac{r}{2}}}\ \sin(k\tfrac{r}{2})$$

$$J_{\frac{3}{2}}(k\tfrac{r}{2}) = \sqrt{\frac{2}{\pi k\frac{r}{2}}}\ \Big(\frac{\sin k\frac{r}{2}}{k\frac{r}{2}} - \cos k\tfrac{r}{2} \Big).$$

Then the first term with $l' = 0$ in the expansion (15) becomes

$$1 - \frac{(k\frac{r}{2})^2}{6} + \frac{(k\frac{r}{2})^4}{120} - \cdots \tag{16a}$$

and the second term with $l' = 1$

$$i \cos\theta \left[k\frac{r}{2} - \frac{1}{10}(k\frac{r}{2})^3 + \cdots \right]. \tag{16b}$$

So we are led to conclusion that the matrix element of the first term in eq. (6) with $l' = 0$ and the matrix element of the second term in eq.(6) with $e^{i\vec{k}\cdot\frac{\vec{r}}{2}}$ replaced by 1, which itself, corresponds to the contribution of electric dipole transition from Guth and Marshall's point of view, differ by something of dimension of octupole contribution, if we neglect higher order terms of $k\frac{r}{2}$. As we find that

$$\left|\langle l=1 \right| \int \rho \vec{\xi}\cdot\vec{E}(\vec{\xi})d^3\vec{x} \left| l=0\rangle\right|^2_{Av} = \frac{\pi\hbar\omega}{V} e^2 \frac{1}{6} \left| \int_0^\infty rF_{f1}(r)\sqrt{\frac{\pi}{2k\frac{r}{2}}}\, J_{\frac{1}{2}}(k\frac{r}{2})F_i(r)dr \right|^2 \tag{17}$$

and
$$\left|\langle l=1 \right| \int \rho \vec{\xi}\cdot\vec{E}(0)d^3\vec{x} \left| l=0\rangle\right|^2_{Av} = \frac{\pi\hbar\omega}{V} e^2 \frac{1}{6} \left| \int_0^\infty rF_{f1}(r)\, F_i(r)dr \right|^2. \tag{18}$$

are substantially different, we see that higher order terms of $k\frac{r}{2}$ are not really small. In eqs. (17) and (18) $F_i(r)$ is the radial wave function divided by r for the ground state; F_{fl} is that for the state of angular momentum l, normalized in the spherical volume V of radius $\mathcal{L}$, in which the photon wave and the nucleonic system are enclosed, as we have indicated before.

So here

$$u_i(\vec{r}) = \frac{F_i(r)}{r} Y_{00} \tag{19a}$$

$$u_{fl}(\vec{r}) = \frac{F_{fl}(r)}{r} Y_{lm} \tag{19b}$$

$$Y_{lm} = \left\{ \frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \right\}^{1/2} P_l^{|m|}(\cos\theta)\, e^{im\varphi} \tag{19c}$$

According to eq. (14), the average square of the matrix element for the transition (II) is

$$\left|\langle l=1|W^E|l=0\rangle\right|^2_{Av} = \left| \frac{1}{2}\left\{ \langle l=1 \right| \int \rho\vec{\xi}\cdot\vec{E}(\vec{\xi})d^3\vec{x} \left| l=0\rangle + \langle l=1 \right| \int \rho\vec{\xi}\cdot\vec{E}(0)d^3\vec{x} \left| l=0\rangle \right\}\right|^2_{Av}$$
$$= \frac{\pi\hbar\omega}{V} e^2 \frac{1}{24} \left| \int_0^\infty rF_{f1}(r)\sqrt{\frac{\pi}{2k\frac{r}{2}}}\, J_{\frac{1}{2}}(k\frac{r}{2})F_i(r)dr + \int_0^\infty rF_{f1}(r)F_i(r)dr \right|^2 \tag{20}$$

The density of states of the final state of the system per unit energy is

$$\rho(E) = \frac{1}{2\pi\hbar}\sqrt{\frac{M}{E_f}}\,\mathcal{L}, \tag{21}$$

where E_f is the final energy of the disintegrated deuteron, and $\mathcal{L}$ is, again, the radius of the sphere of volume V, in which the photon wave is enclosed and in which now the nucleon's wave function after the disintegration is normalized. Then for transition (II), the cross-section is given by

$$\sigma_{\ell=1} = \frac{\pi}{24}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\frac{\hbar ck}{\hbar c}\left|\sqrt{\mathcal{L}}\left\{\int_0^\infty rF_{f1}(r)\sqrt{\frac{\pi}{2k\frac{r}{2}}}J_{\frac{1}{2}}\left(k\frac{r}{2}\right)F_i(r)\,dr + \int_0^\infty rF_{f1}(r)F_i(r)\,dr\right\}\right|^2 \tag{22}$$

As we mentioned before that eq. (18) alone is the average square of the matrix element for the electric dipole transition from Guth and Marshall's point of view, we see that the cross-section for electric dipole transition considered by Guth and Marshall is

$$\sigma_d = \frac{\pi}{6}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\frac{\hbar ck}{\hbar c}\left|\sqrt{\mathcal{L}}\int_0^\infty rF_f(r)\,F_i(r)\,dr\right|^2 \tag{23}$$

The value of the cross-sections for $\sigma_{\ell=1}$ and σ_d for two effective ranges 1.56×10^{-13} cm. and 1.74×10^{-13} cm. and for the energy of the incident photon ranging from 100 Mev to 150 Mev will be given in Section 3.

For the transition (III), the average square of the matrix element of the first two terms in eq. (6) is

$$\left|\langle \ell=2|W^E|\ell=0\rangle\right|^2_{Av} = \left|\frac{1}{2}\left\{\langle \ell=2|\int \rho\vec{\xi}\cdot\vec{E}(\vec{\xi})\,d^3\vec{x}|\ell=0\rangle + \langle \ell=2|\int \rho\vec{\xi}\cdot\vec{E}(0)\,d^3\vec{x}|\ell=0\rangle\right\}\right|^2_{Av} \tag{24}$$

As the second term in the absolute square vanishes, this gives

$$\begin{aligned}\left|\langle \ell=2|W^E|\ell=0\rangle\right|^2_{Av} &= \frac{2\pi\hbar\omega e^2}{V}\left|\frac{1}{2}\int u^*_{f2}(\vec{r})\;\vec{e}_k\cdot\frac{\vec{r}}{2}\;e^{i\vec{k}\cdot\frac{\vec{r}}{2}}\;u_i(\vec{r})\,d^3\vec{r}\right|^2_{Av} \\ &= \frac{\pi\hbar\omega e^2}{V}\frac{1}{30}\left|\frac{3}{2}\int_0^\infty rF_{f2}(r)\sqrt{\frac{\pi}{2k\frac{r}{2}}}J_{\frac{3}{2}}\left(k\frac{r}{2}\right)F_i(r)\,dr\right|^2\end{aligned} \tag{25}$$

and the cross-section is then given by

$$\sigma_{\ell=2} = \frac{3\pi}{40}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\frac{\hbar ck}{\hbar c}\left|\sqrt{\mathcal{L}}\int_0^\infty rF_{f2}(r)\sqrt{\frac{\pi}{2k\frac{r}{2}}}J_{\frac{3}{2}}\left(k\frac{r}{2}\right)F_i(r)\,dr\right|^2 \tag{26}$$

If we want to compare the result (25) with the cross-section for the electric quadrupole transition calculated by the method of Møller and Rosenfeld[19], we remark that they used for it the matrix element

$$- (Q.\vec{\nabla}_o)\vec{E}_o = -\frac{1}{2}\int \sum_i \sum_j \rho \xi_i \xi_j (\nabla_j E_i)_o \, d^3\vec{x} \tag{27}$$

where $\vec{\nabla}_o . \vec{E}_o = (\vec{\nabla} . \vec{E})_o$ stands for the value of $\vec{\nabla}.\vec{E}$ in the center of gravity of the nucleonic system. In Section 4, we show that this leads to a result similar to the second member of eq. (25) but with $e^{i\vec{k}.\vec{r}/2}$ replaced by $i\vec{k}.\frac{\vec{r}}{2}$ that is, the contribution of the second term of a Taylor's expansion of $e^{i\vec{k}.\frac{\vec{r}}{2}}$. (The first term, which is 1, in the Taylor's expansion of $e^{i\vec{k}.\frac{\vec{r}}{2}}$ does not contribute anything to the integral in the second member of eq. (25). Now in the last member of eq.(25) we expand $e^{i\vec{k}.\frac{\vec{r}}{2}}$ according to the expansion (15), but neglect terms with $\ell' \geq 2$. That is, only the term $\ell' = 1$ need be taken into account. This means replacing $e^{i\vec{k}.\vec{r}/2}$ by (16b), so that the difference between the result calculated by using the last member of eq.(25) and the result obtained by the Møller and Rosenfeld method (which was used by Guth and Marshall for the calculation of the cross-section for electric quadrupole transitions using exponential and Hulthen potentials) is something of the dimension of higher multipole contributions. The cross-section for the electric quadrupole transition by the method of Møller and Rosenfeld is then given by

$$\sigma_q = \frac{\pi}{120}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\frac{\hbar ck}{\hbar c}\left|\sqrt{\pi}\int_0^\infty rF_{f2}(r)\, k\frac{r}{2}\, F_i(r)dr\right|^2 \tag{28}$$

In fact, we find for $\sigma_{\ell=2}$ and σ_q, as calculated according to eqs (26) and (28) for the effective range 1.56×10^{-13}cm. and for an energy of 100 Mev for the incident γ- ray,

$$\sigma_{\ell=2} = 1.68 \times 10^{-30}\text{cm}^2.$$

and $$\sigma_q = 1.02 \times 10^{-30}\text{cm}^2.$$

These results show that higher order terms of $k\frac{r}{2}$ are not quite small. As $\sigma_{\ell=2}$ is small compared with the uncertainty caused by neglecting higher order effects (comparing the results of σ_d with those of $\sigma_{\ell=1}$ in Section 3), we did not take the trouble of further calculation of $\sigma_{\ell=2}$ for other energies of the incident γ-ray, or for another effective range. But we calculate σ_q for the two different ranges, i.e. 1.56×10^{-13}cm. and 1.74×10^{-13}cm., and for the energy of the incident γ-ray ranging from 100 Mev to 150 Mev just for the sake of comparison with the results calculated by Guth and Marshall for different kinds of potentials.

As we have stated before, the magnetic quadrupole contribution cannot be calculated, because we do not know the distribution of the magnetization of the deuteron over space, that is, we do not know $\vec{M}$ as a function of $\vec{\xi}$, but we only know $\int \vec{M}\, d^3\vec{x}$ from experiments on magnetic moments. Nevertheless if we still want to calculate the cross-section for magnetic quadrupole transitions, we should note that the magnetic quadrupole terms in the matrix element are only two-thirds of the contribution of the second term of a Taylor expansion of $e^{i\vec{k}\cdot\vec{\xi}}$ in the expression for $\vec{H}(\vec{\xi})$ in $-\int \vec{M}\cdot\vec{H}(\vec{\xi})\, d^3\vec{x}$, as shown in section 4. (This is because there are also magnetic quadrupole contributions from the last term in eq. (3)).

We have calculated the magnetic quadrupole cross-section (hoping the best as for eq.(9)) for two different ranges, and for energies of the incident γ-~~ray~~ quantum ranging from 100 Mev to 150 Mev. In how far this is exactly equivalent to what Guth and Marshall have done for different kinds of potentials, is not clear, as we have used the equation

$$\sigma_{mq} = \frac{\pi}{54}\frac{e^2}{\hbar c}\sqrt{\frac{Mc^2}{E_f}}\frac{(\hbar ck)^2}{(Mc^2)^2}\frac{\hbar ck}{\hbar c}\left(\mu_p^2+\mu_N^2-\frac{2}{3}\mu_p\mu_N\right)\left|\sqrt{\alpha}\int_0^\infty rF_{f1}(r)F_i(r)\,dr\right|^2$$

$$= \frac{1}{9}\left(\frac{E_\gamma}{Mc^2}\right)^2\left(\mu_p^2+\mu_N^2-\frac{2}{3}\mu_p\mu_N\right)\sigma_d\,, \qquad (29)$$

where $E_\gamma = \hbar ck$ is the energy of the incident phonton, while Guth and Marshall found a factor $\frac{1}{6}$ instead of $\frac{1}{9}$ in the last equation, and used $(\mu_p^2 + \mu_N^2 - 2\mu_p\mu_N)$ instead of the expression $(\mu_p^2 + \mu_N^2 - \frac{2}{3}\mu_p\mu_N)$ in our equation.

The derivations of these expressions for the cross-sections will be given in section 4.

3. NUMERICAL RESULTS AND GRAPHS

Table I gives the calculated values of the cross-section $\sigma_{\ell=1}$ for the electric transition (II).* Table II gives the values of the cross-sections σ_d and σ_q for the electric dipole and quadrupole transitions calculated following Møller and Rosenfeld's method which was used by Guth and Marshall for calculating σ_d and σ_q by using exponential and Hulthen potentials. Table III gives the values of the cross-section σ_{mq} for the magnetic quadrupole transition. The values given in Table III are listed here for the sake of comparison with Guth and Marshall's results, although their inclusion, according to our previous discussion, does not have a sound theoretical basis.

Table I

r_o (effective range) $\times 10^{13}$ cm.	1.56	E_γ (energy of the incident -ray) in Mev	100	125	150
		$\sigma_{\ell=1} \times 10^{30}\,cm^2$	58.23	40.24	26.84
$r_o \times 10^{13}$ cm.	1.74	E_γ in Mev	100	125	150
		$\sigma_{\ell=1} \times 10^{30}\,cm^2$	53.33	37.50	24.24

Table II

$r_o \times 10^{13}$ cm.	1.56	E_γ in Mev	100	125	150
		$\sigma_d \times 10^{30}\,cm^2$	44.81	29.11	16.72
		$\sigma_q \times 10^{30}\,cm^2$	1.02	0.84	0.61
$r_o \times 10^{13}$ cm.	1.74	E_γ in Mev	100	125	150
		$\sigma_d \times 10^{30}\,cm^2$	36.44	25.80	16.45
		$\sigma_q \times 10^{30}\,cm^2$	0.91	0.65	0.55

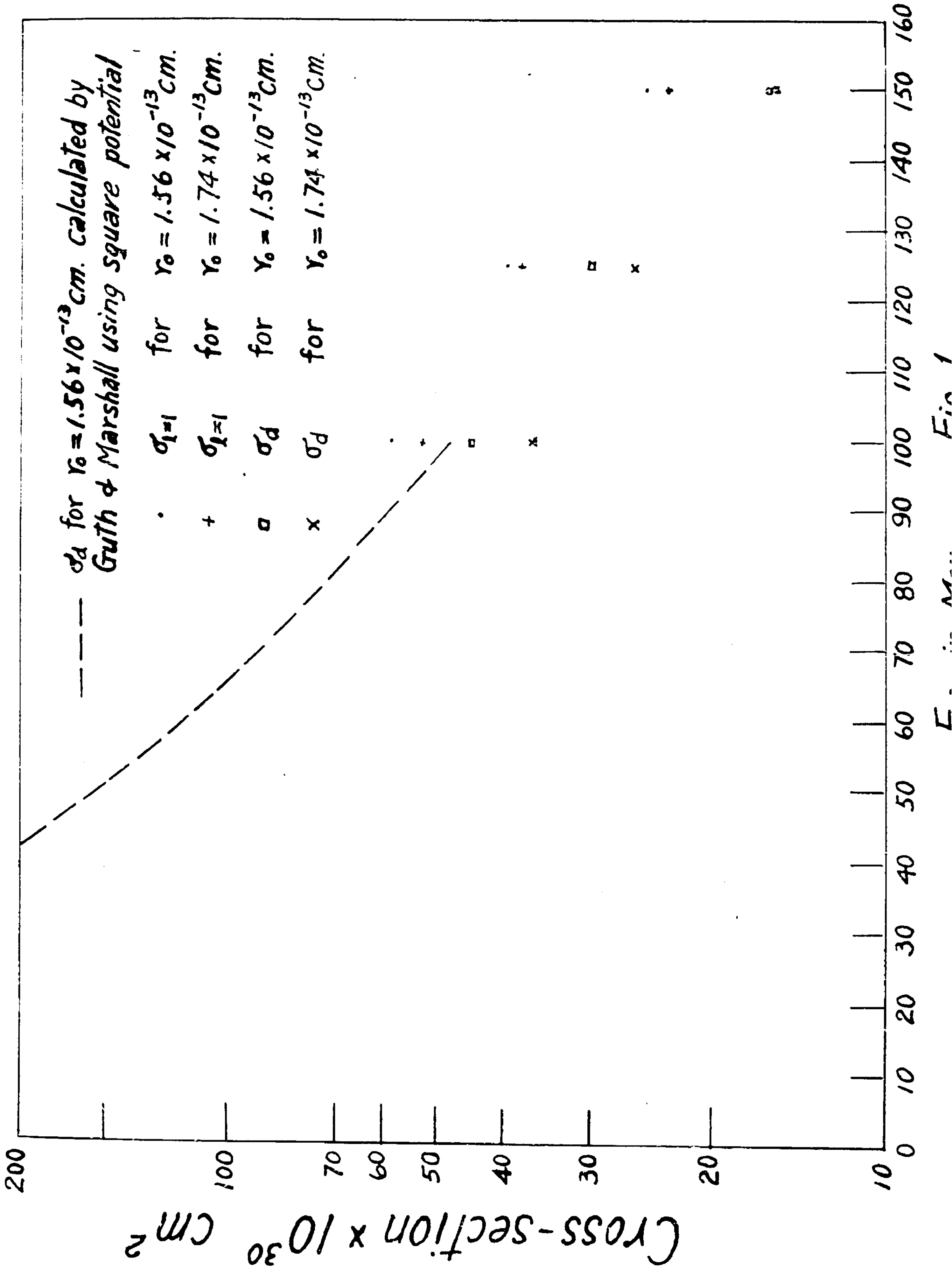
*See page 7.

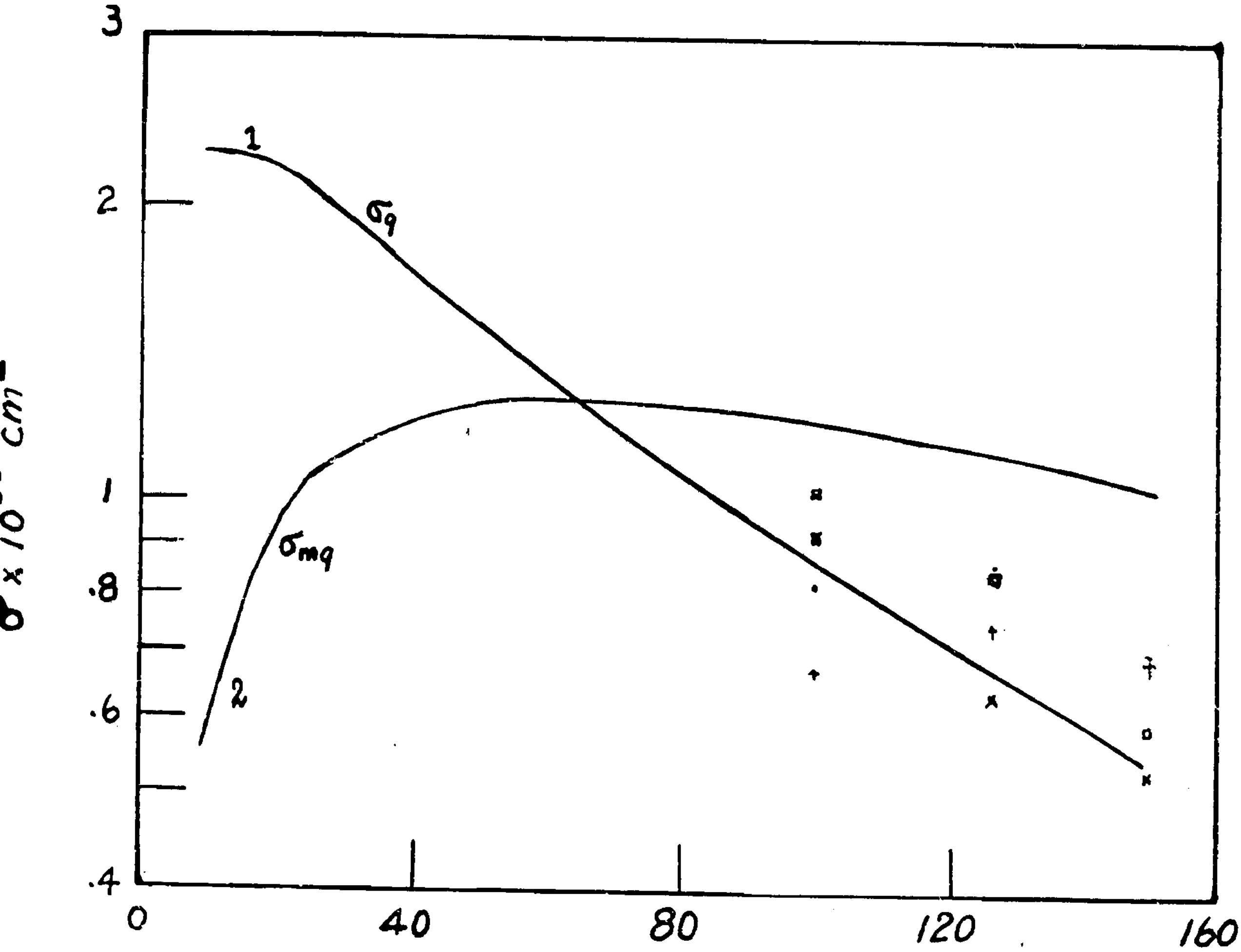
Table III

$r_o \times 10^{13}$cm.	1.56	E_γ in Mev	100	125	150
		$\sigma_{mq} \times 10^{30}\text{cm}^2$	0.83	0.85	0.71
$r_o \times 10^{13}$cm.	1.74	E_γ in Mev	100	125	150
		$\sigma_{mq} \times 10^{30}\text{cm}^2$	0.68	0.76	0.70

As we can see from Tables I and II that the values of $\sigma_{\ell=1}$ and σ_d are substantially different, higher order terms of $k\frac{r}{2}$ are not quite negligible.

On the following two pages there are graphs showing these results. Compare, however, the preceding section as to the comparison of σ_{mq} as calculated by us and as calculated by Guth and Marshall.


σd for r0 = 1.56 x 10^-13 cm. calculated by Guth & Marshall using square potential
• σl=1 for r0 = 1.56 x 10^-13 cm.
+ σl=1 for r0 = 1.74 x 10^-13 cm.
□ σd for r0 = 1.56 x 10^-13 cm.
x σd for r0 = 1.74 x 10^-13 cm.
Cross-section x 10^30 cm^2
200
100
70
60
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160



E_γ in Mev

Fig. 2

1 and 2 calculated by Guth & Marshall using Hulthen potential for $r_0 = 1.74 \times 10^{-13}$ cm.

•	σ_{mq}	for	$r_0 = 1.56 \times 10^{-13}$ cm.
+	σ_{mq}	for	$r_0 = 1.74 \times 10^{-13}$ cm.
□	σ_q	for	$r_0 = 1.56 \times 10^{-13}$ cm.
x	σ_q	for	$r_0 = 1.74 \times 10^{-13}$ cm.

4. MATHEMATICAL APPENDIX

As we have stated before, in this section we shall give derivations of all the important formulas appearing in the text. We shall first consider eq.(3). The perturbation used in our problem is

$$W = -\int \vec{I} \cdot \vec{A} \, d^3\vec{x} \ , \tag{30}$$

which is integrated over the current field. Here $\vec{A}$ is the transverse part of the vector potential of the photon field. We can write eq.(30) as

$$W = -\sum_{ij}\int I_i A_j \ (\nabla_i \xi_j) \ d^3\vec{x} = \sum_{ij}\int \xi_j (I_i \nabla_i A_j + A_j \nabla_i I_i \ ) \ d^3\vec{x} \ , \tag{31}$$

where $\vec{\xi}$ has the same meaning as stated before. By the equation of continuity,

$$\frac{1}{c}\dot{\rho} + \vec{\nabla} \cdot \vec{I} = 0 \ ,$$

we have $W = \sum_{ij}\int \xi_j I_i \nabla_i A_j d^3\vec{x} - \frac{1}{c}\int \dot{\rho} \ (\vec{\xi} \cdot \vec{A}) \ d^3\vec{x}$

$$= \sum_{ij}\int \xi_j I_i \nabla_i A_j d^3\vec{x} - \frac{1}{c}\frac{d}{dt}\int \rho (\vec{\xi} \cdot \vec{A}) d^3\vec{x} + \frac{1}{c}\int \rho (\vec{\xi} \cdot \dot{\vec{A}}) \ d^3\vec{x} \ .$$

Since $\vec{A}$ is the transverse part, it follows from

$$\vec{E} = -\frac{1}{c}\dot{\vec{A}} \tag{32}$$

where $\vec{E}$ is the electric photon field, that

$$W = -\frac{1}{c}\frac{d}{dt}\int \rho \ (\vec{\xi} \cdot \vec{A}) d^3\vec{x} - \int \rho \ (\vec{\xi} \cdot \vec{E}) d^3\vec{x} + \sum_{ij}\int \xi_j I_i \nabla_i A_j d^3\vec{x}.$$

We can leave the first term out of consideration, since its matrix elements are zero between two states with the same total energy (including the photon energy) and, consequently, it does not give any contribution to the matrix element of W considered in our first order perturbation. So, if for the sake of simplicity, we just still use the same notation W for the altered interaction operator, we can write

$$W = -\int \rho \ (\vec{\xi} \cdot \vec{E}(\vec{\xi}) \ ) d^3x + \sum_{ij}\int \xi_j I_i \nabla_i A_j d^3\vec{x}. \tag{33}$$

We shall now show that the second term stands partially for the magnetic interaction

$$-\int \vec{M} \cdot \vec{H} \ d^3\vec{x} \ , \tag{34}$$

where $\vec{H}$ is the magnetic photon field and where we consider

$$\vec{M} = \frac{1}{2}\vec{\xi} \times \vec{I} \tag{35}$$

as the "magnetization" of the deuteron. Indeed,

$$-\int \vec{M}\cdot\vec{H}\, d^3\vec{x} = -\frac{1}{2}\int (\vec{\xi}\times\vec{I})\cdot(\vec{\nabla}\times\vec{A})\, d^3\vec{x}$$
$$= -\frac{1}{2}\sum_{ij}\int \xi_i I_j (\nabla_i A_j) d^3\vec{x} + \frac{1}{2}\sum_{ij}\int \xi_j I_i (\nabla_i A_j) d^3x \tag{36}$$

Subtracting eq.(36) from eq.(33) and the adding eq.(34) to both members of the equation, we find

$$W = -\int \rho(\vec{\xi}\cdot\vec{E}(\vec{\xi}))\, d^3\vec{x} - \int \vec{M}\cdot\vec{H}\, d^3\vec{x} + \frac{1}{2}\sum_{ij}\int (\xi_j I_i + \xi_i I_j)\, \nabla_i A_j d^3\vec{x}\,, \tag{37}$$

in which, by $\nabla_i \xi_j = \delta_{ij}$ and by integration by parts, the last term is equal to

$$\frac{1}{2}\sum_{ij}\int (\xi_j I_i + \xi_i I_j)\nabla_i A_j d^3\vec{x} = \frac{1}{2}\sum_{ijk}\int \left\{ I_k \nabla_k(\xi_i \xi_j)\right\} (\nabla_i A_j) d^3\vec{x}$$
$$= -\frac{1}{2}\sum_{ijk}\int \xi_i\xi_j (\nabla_k I_k)(\nabla_i A_j) d^3\vec{x} - \frac{1}{2}\sum_{ijk}\int \xi_i \xi_j I_k (\nabla_k \nabla_i A_j) d^3\vec{x}. \tag{38a}$$

Again, using the equation of continuity, the first term becomes

$$-\frac{1}{2}\sum_{ijk}\int \xi_i \xi_j (\nabla_k I_k)(\nabla_i A_j) d^3\vec{x} = \frac{1}{2}\sum_{ij}\int \frac{\dot{\rho}}{c}\xi_i\xi_j(\nabla_i A_j) d^3\vec{x}$$
$$= \frac{d}{dt}\frac{1}{2}\sum_{ij}\int \frac{\rho}{c}\xi_i\xi_j(\nabla_i A_j) d^3\vec{x} - \frac{1}{2}\sum_{ij}\int \rho\,\xi_i\xi_j\left(\nabla_i \frac{\dot{A}_j}{c}\right) d^3\vec{x} \tag{38b}$$

Substituting eqs.(38a) and (38b) together with eq. (32) into eq. (37), and again, leaving out of consideration the one but last term in (38b) by the same argument as before, we get, finally,

$$W = -\int \rho\vec{\xi}\cdot\vec{E}(\vec{\xi}) d^3\vec{x} - \int \vec{M}\cdot\vec{H}\, d^3\vec{x} + \frac{1}{2}\sum_{ij}\int \rho\,\xi_i\xi_j(\nabla_i E_j) d^3\vec{x} - \frac{1}{2}\sum_{ijk}\int \xi_i\xi_j I_k(\nabla_k\nabla_i A_j) d^3\vec{x}$$

which is eq. (3) in section 2.

In eq.(4) (in section 2) we introduced $\vec{E}(0)$, which leads to eq.(6), in which all electric quadrupole terms were taken together. In eq.(3) they appear partially in $-\int \rho\vec{\xi}\cdot\vec{E}(\vec{\xi}) d^3\vec{x}$ and partially in $\frac{1}{2}\sum_{ij}\int \rho\,\xi_i\xi_j(\nabla_i E_j) d^3\vec{x}$.

Going a term further in the expansion, we could have written

$$\sum_i \int \rho \xi_i E_i(0) d^3\vec{x} = \sum_i \int \rho \xi_i E_i(\vec{\xi}) d^3\vec{x} - \sum_{ij} \int \rho \xi_i \xi_j (\nabla_i E_j) d^3\vec{x}$$
$$+ \frac{1}{2} \sum_{ijk} \int \rho \xi_i \xi_j \xi_k (\nabla_i \nabla_j E_k) d^3\vec{x} - \ldots \qquad (39a)$$

Thence we find W given by

$$W = -\frac{1}{2} \sum_i \int \rho \xi_i E_i(\vec{\xi}) d^3\vec{x} - \frac{1}{2} \sum_i \int \rho \xi_i E_i(0) d^3\vec{x} + \frac{1}{4} \sum_{ijk} \int \rho \xi_i \xi_j \xi_k (\nabla_i \nabla_j E_k) d^3\vec{x}$$
$$- \int \vec{M} \cdot \vec{H}(\vec{\xi}) d^3\vec{x} - \frac{1}{2} \sum_{ijk} \int \xi_i \xi_j I_k (\nabla_k \nabla_i A_j) d^3\vec{x} \qquad (39b)$$

which gives eq.(6) when we neglect the electric (octupole and magnetic) quadrupole terms. Eq.(39b) contains magnetic quadrupole contributions from both last two terms. In order to take this contribution together, we proceed by a similar kind of reasoning as used in eq. (39a) for the electric dipole term, where we took together the electric quadrupole contributions by comparing $\int \rho \vec{\xi} \cdot \vec{E}(\vec{\xi}) d^3\vec{x}$ with $\int \rho \vec{\xi} \cdot \vec{E}(0) d^3\vec{x}$. Here, we consider

$$\int \vec{M} \cdot \vec{H}(0) d^3\vec{x} = \int \vec{M} \cdot \vec{H}(\vec{\xi}) d^3\vec{x} - \int \vec{M} \cdot \sum_k \xi_k \nabla_k \vec{H}\, d^3\vec{x} - \ldots$$
$$= \int \vec{M} \cdot \vec{H}(\vec{\xi}) d^3\vec{x} - \frac{1}{4} \sum_{ijk} \int (\xi_k I_j - \xi_j I_k) \xi_i \nabla_i (\nabla_k A_j - \nabla_j A_k) d^3\vec{x} - \ldots \qquad (39c)$$
$$= \int \vec{M} \cdot \vec{H}(\vec{\xi}) d^3\vec{x} - \frac{1}{2} \sum_{ijk} \int \xi_i \xi_k I_j (\nabla_k \nabla_i A_j) + \frac{1}{2} \sum_i \sum_j \sum_k \int \xi_i \xi_j I_k (\nabla_k \nabla_i A_j) d^3\vec{x} - \ldots$$

The first and the third terms here are just minus the last two terms of eq.(39b), so that we also write

$$W = -\sum_i \int \rho \xi_i \frac{E_i(\vec{\xi}) + E_i(0)}{2} d^3\vec{x} + \frac{1}{4} \sum_{ijk} \int \rho \xi_i \xi_j \xi_k (\nabla_i \nabla_j E_k) d^3\vec{x}$$
$$- \int \vec{M} \cdot \vec{H}(0) d^3\vec{x} - \frac{1}{2} \sum_{ijk} \int \xi_i \xi_k I_j (\nabla_k \nabla_i A_j) d^3\vec{x} + \ldots \qquad (39d)$$

We may also take for W a kind of an average calculated as

$$W = \frac{2}{3} W(39b) + \frac{1}{3} W(39d)$$
$$= -\sum_i \int \rho \xi_i \frac{E_i(\vec{\xi}) + E_i(0)}{2} d^3\vec{x} + \frac{1}{4} \sum_{ijk} \int \rho \xi_i \xi_j \xi_k (\nabla_i \nabla_j E_k) d^3\vec{x}$$
$$- \int \vec{M} \cdot \frac{2\vec{H}(\vec{\xi}) + \vec{H}(0)}{3} d^3\vec{x} - \frac{1}{6} \sum_{ijk} \int (2\xi_i \xi_j I_k + \xi_i \xi_k I_j)(\nabla_k \nabla_i A_j) d^3\vec{x} + \ldots \qquad (39e)$$

We shall now show that the last sum in eq. (39e) contains only octupole and higher multipole terms. For this purpose, we write

$$+\frac{1}{6}\sum_{ijk}\int(2\xi_i\xi_j I_k+\xi_i\xi_k I_j)(\nabla_k\nabla_i A_j)\,d^3\vec{x}$$

$$=+\frac{1}{6}\sum_i\sum_j\sum_k\int(\xi_i\xi_j I_k+\xi_j\xi_k I_i+\xi_k\xi_i I_j)(\nabla_k\nabla_i A_j)\,d^3\vec{x}$$

$$=+\frac{1}{6}\sum_{ijk\ell}\int(\xi_i\xi_j\delta_{k\ell}+\delta_{i\ell}\xi_j\xi_k+\xi_i\delta_{j\ell}\xi_k)I_\ell\,(\nabla_k\nabla_i A_j)\,d^3\vec{x}$$

$$=+\frac{1}{6}\sum_{ijk\ell}\int\left\{\nabla_\ell(\xi_i\xi_j\xi_k)\right\}I_\ell\left(\nabla_k\nabla_i A_j\right)d^3\vec{x} \qquad (39f)$$

Integrating this by parts over x_ℓ gives

$$-\frac{1}{6}\sum_{ijk\ell}\int\left\{\xi_i\xi_j\xi_k\,(\nabla_\ell I_\ell)(\nabla_k\nabla_i A_j)+\xi_i\xi_j\xi_k I_\ell\,(\nabla_i\nabla_k\nabla_\ell A_j)\right\}d^3\vec{x} \qquad (39g)$$

Now using $\sum_\ell\nabla_\ell I_\ell=-\frac{\dot{\rho}}{c}$, and separating a term which is pure time derivative, we find for this

$$\frac{1}{6c}\frac{d}{dt}\sum_{ijk}\int\rho\xi_i\xi_j\xi_k\,(\nabla_k\nabla_i A_j)\,d^3\vec{x}-\frac{1}{6}\sum_{ijk}\int\rho\xi_i\xi_j\xi_k\left(\nabla_k\nabla_i\frac{\dot{A}_j}{c}\right)d^3\vec{x}$$
$$-\frac{1}{6}\sum_{ijk\ell}\int\xi_i\xi_j\xi_k I_\ell\,(\nabla_i\nabla_k\nabla_\ell A_j)\,d^3\vec{x}$$

$$=\frac{1}{6c}\frac{d}{dt}\sum_{ijk}\int\rho\xi_i\xi_j\xi_k\,(\nabla_k\nabla_i A_j)\,d^3\vec{x}+\frac{1}{6}\sum_{ijk}\int\rho\xi_i\xi_j\xi_k\,(\nabla_k\nabla_i E_j)\,d^3\vec{x}$$
$$-\frac{1}{6}\sum_{ijk\ell}\int\xi_i\xi_j\xi_k I_\ell\,(\nabla_i\nabla_k\nabla_\ell A_j)\,d^3\vec{x} \qquad (39h)$$

Thence, omitting the time derivative again for W, we find, by subtracting eq. (39h) for the last sum in (39e),

$$W=-\int\rho\vec{\xi}\cdot\frac{\vec{E}(\vec{\xi})+\vec{E}(0)}{2}\,d^3\vec{x}-\int\vec{M}\cdot\frac{2\vec{H}(\vec{\xi})+\vec{H}(0)}{3}+\sum_{ijk}\int\rho\xi_i\xi_j\xi_k\left(\frac{1}{4}\nabla_i\nabla_j E_k-\frac{1}{6}\nabla_k\nabla_i E_j\right)d^3\vec{x}$$
$$+\frac{1}{6}\sum_{ijk\ell}\int\xi_i\xi_j\xi_k I_\ell\,(\nabla_i\nabla_k\nabla_\ell A_j)\,d^3\vec{x}+\ldots, \qquad (39i)$$

where, for the sake of simplicity the notation W is still used after omitting the time derivative term. Here the sum over i, j, k is of dimension of the electric octupole interaction, while the sum over i, j, k, ℓ is of the dimension of the magnetic octupole interaction. Thence, including only up to electric and magnetic quadrupole terms, we find

$$W = W^E + W^M \tag{39j}$$

with

$$W^E = -\int \rho \vec{\xi} \cdot \frac{\vec{E}(\vec{\xi}) + \vec{E}(0)}{2}\, d^3\vec{x} \tag{39k}$$

$$W^M = -\int \vec{M} \cdot \frac{2\vec{H}(\vec{\xi}) + \vec{H}(0)}{3}\, d^3\vec{x} \tag{39l}$$

Eq.(39k) shows that the electric quadrupole term is only half the contribution from the second term in the Taylor's expansion for $e^{i\vec{k}\cdot\frac{\vec{\xi}}{2}}$ in $\vec{E}(\vec{\xi})$ appearing in $-\int \rho \vec{\xi} \cdot \vec{E}(\vec{\xi}) d^3\vec{x}$, while (39l) shows that the magnetic quadrupole term is only two-thirds of the contribution from the second term in the expansion for $e^{i\vec{k}\cdot\frac{\vec{\xi}}{2}}$ in $\vec{H}(\vec{\xi})$ in $-\int \vec{M} \cdot \vec{H}(\vec{\xi}) d^3\vec{x}$.

In order to derive the remaining important equations, we first find the wave functions of the ground state of the deuteron and of the disintegrated state of the deuteron after the absorption of the photon. On assuming the 50-50 mixture of ordinary and Majorana forces, the Hamiltonian of the two nucleons is then given by

$$H = \frac{1}{2M} P_1^2 + \frac{1}{2M} P_2^2 + V(\vec{x}_2 - \vec{x}_1) \frac{1}{2} \left[1 - \frac{1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2}{2} \; \frac{1 + \vec{\tau}_1 \cdot \vec{\tau}_2}{2} \right], \tag{40}$$

in which the subscripts 1 and 2 referred to the two nucleons, i.e. proton and neutron, respectively, and $\vec{\sigma}$, $\vec{\tau}$ are the "spin" and "isotopic spin" operators. Let σ be a one particle "spin" coordinate, which takes the values $+1$ and -1, and let τ be the "isotopic spin" coordinate, which takes the value $+1$ for a proton state and -1 for a neutron state. Then the stationary state wave equation is

$$\left(-\frac{\hbar^2}{2M} \nabla_1^{\,2} - \frac{\hbar^2}{2M} \nabla_2^{\,2}\right) \Psi(\vec{x}_1, \vec{x}_2; \sigma_1, \sigma_2; \tau_1, \tau_2) + V(\vec{x}_2 - \vec{x}_1) \frac{1}{2} \left[\Psi(\vec{x}_1, \vec{x}_2; \sigma_1, \sigma_2; \tau_1, \tau_2) - \Psi(\vec{x}_1, \vec{x}_2; \sigma_2, \sigma_1; \tau_2, \tau_1) \right]$$

$$= E_{tot} \Psi(\vec{x}_1, \vec{x}_2; \sigma_1, \sigma_2; \tau_1, \tau_2) \tag{41}$$

Let $\vec{r} = \vec{x}_2 - \vec{x}_1$ (42a)

and $M_c = 2M$ $\quad M_c\vec{R} = M(\vec{x}_1 + \vec{x}_2)$. (42b)

Then $-\frac{\hbar^2}{2M}\nabla_1^2 - \frac{\hbar^2}{2M}\nabla_2^2 = -\frac{\hbar^2}{2Mc}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2$ (42c)

where $\mu = M/2$ is the reduced mass. We now attempt to separate the wave equation by taking $\Psi(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1\tau_2)$ as the product of a function of $\vec{R}$ and a function of $\vec{r}$; $\sigma_1,\sigma_2;\tau_1\tau_2$

writing $\Psi(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1\tau_2) = U(\vec{R})\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2)$.

Then we get $-\frac{\hbar^2}{2Mc}\nabla_R^2 U(\vec{R}) = E_c U(\vec{R})$ (41a)

and
$$-\frac{\hbar^2}{2\mu}\nabla_r^2\psi(\vec{r};\sigma_1,\sigma_2;\tau_1\tau_2) + V(\vec{r})\frac{1}{2}\left[\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) - \psi(\vec{r};\sigma_2,\sigma_1;\tau_2,\tau_1)\right]$$
$$= (E_{tot} - E_c)\,\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) \qquad (41b)$$

in which E_c is the kinetic energy due to the motion of the nucleonic system. Let E be the energy of the system aside from this kinetic energy E_c, i.e. $E = E_{tot} - E_c$. Then eq.(41b) becomes
$$-\frac{\hbar^2}{M}\nabla_r^2\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) + V(\vec{r})\frac{1}{2}\left[\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) - (\vec{r};\sigma_2,\sigma_1;\tau_2,\tau_1)\right)$$
$$= E\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) \qquad (41c)$$

because $\mu = M/2$. According to Pauli's exclusion principle, the wave function must be antisymmetric with respect to the interchange of all coordinates of the two nucleons: i.e.
$$\psi(\vec{r};\sigma_2,\sigma_1;\tau_2,\tau_1) = -\psi(-\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2). \qquad (43a)$$

But we know that the wave function has the parity l, where l is the quantum number for the angular momentum; i.e.
$$\psi(-\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) = (-1)^l\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2). \qquad (43b)$$

Substituting eqs.(43a) and (43b) into eq. (41c), we find
$$-\frac{h^2}{M}\nabla_r^2\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) + \frac{1}{2}\left[1+(-1)^l\right]V(\vec{r})\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) = E\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) \quad (41d)$$

Let us write the wave function in the form

$$\psi(\vec{r};\sigma_1,\sigma_2;\tau_1,\tau_2) = u(\vec{r})\,\chi_S^{m_s}(\sigma_1,\sigma_2)\,\zeta_T^{m_T}(\tau_1,\tau_2), \qquad (44)$$

where S and m_s are the quantum numbers for the total spin of the two nucleons and for its component along the z-direction, and T and m_T the corresponding quantum numbers for the isotopic spin. For the symmetric triplet state $S=1$, $m_s = -1, 0, 1$ and

$$\chi_1^1 = \alpha(1)\,\alpha(2)$$

$$\chi_1^0 = \frac{1}{\sqrt{2}}\left[\alpha(1)\,\beta(2) + \alpha(2)\,\beta(1)\right]$$

$$\chi_1^{-1} = \beta(1)\,\beta(2)$$

and for antisymmetric singlet state, $S = 0$, $m_s = 0$ and

$$\chi_0^0 = \frac{1}{\sqrt{2}}\left[\alpha(1)\,\beta(2) - \alpha(2)\,\beta(1)\right] .$$

Similar equations hold for $\zeta_T^{m_T}$. We can now easily see that eq.(41d) becomes

$$-\frac{\hbar^2}{M}\nabla_r^2 u(\vec{r}) + \frac{1}{2}\left[1 + (-1)^{\ell}\right] V(r)\,u(\vec{r}) = E\,u(\vec{r}) . \qquad (45)$$

in which we have replaced $V(\vec{r})$ by $V(r)$, because we only consider the central field. This is the equation for determining the spatial wave function. If we separate the wave function into radial and angular parts, we get the solution in the usual way, i.e.

$$u(\vec{r}) = \frac{F(r)}{r}\,Y_{\ell,m}(\theta,\varphi) \qquad (46)$$

Substituting eq.(46) back into eq.(45), we then arrive at the well-known equation

$$\frac{d^2F}{dr^2} + \left\{\frac{M}{\hbar^2}\left[E - \frac{1}{2}\left(1 + (-1)^{\ell}\right)V(r)\right] - \frac{\ell(\ell+1)}{r^2}\right\}F = 0 \qquad (47)$$

Now we are going to find the spatial wave functions of the initial and final states of the deuteron. Let its initial state be the ground state; i.e. the ^3S state. Then $\ell = 0$ in the initial state,

and $$u_i(\vec{r}) = \frac{F_i(r)}{r}\,Y_{0,0} = \frac{1}{(4\pi)^{1/2}}\,\frac{F_i(r)}{r} ,$$

and, consequently, eq. (47) becomes

$$\frac{d^2F_i}{dr^2} + \frac{M}{\hbar^2}\Big[E - V(r)\Big] F_i = 0 \tag{48}$$

Let the depth and width (i.e. the effective range) of the square well potential be V_o and r_c respectively. Then

$$V(r) = \begin{cases} -V_o & r < r_c \\ 0 & r > r_c \end{cases} \tag{49}$$

With $E = -W_o$, where $W_o > 0$ is the binding energy of the deuteron in its ground state*, eq.(48) becomes (inside and outside) the potential well

$$\frac{d^2F_i}{dr^2} + \frac{M}{\hbar^2}(V_o - W_o)F_i = 0 \qquad r < r_o \tag{50a}$$

$$\frac{d^2F_i}{dr^2} - \frac{M}{\hbar^2} W_oF_i = 0 \qquad r > r_o \tag{50b}$$

$\frac{F_i(r)}{r}$ must be continuous and bounded and have a continuous derivative everywhere. Therefore, F_i must have the same continuity condition, must go to zero at $r = 0$, and must not diverge faster than r as $r \to \infty$. To satisfy the conditions at origin and infinity the solution of eqs.(50a) and (50b) must be

$$F_i = A_i \sin k_i r \qquad r < r_o \tag{51a}$$

$$F_i = B_i\, e^{-\alpha_i r} \qquad r > r_o \tag{51b}$$

where $k_i = \sqrt{M(V_o - W_o)}/\hbar \qquad \alpha_i = \sqrt{MW_o}/\hbar \qquad (53)$

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* Note that the use of Christian and Hart's spin independent mixture of ordinary and Majorana forces leads here to the same binding energy in ^1S state as in the ^3S state, contrary to the experimental evidence. This is one of the reasons why Christian and Hart's assumption cannot be regarded as well based. See section 1.

in which the subscript i indicates the quantities for the initial state.

Since F_i must be continuous everywhere, applying this at $r = r_o$ gives

$$A_i \sin \kappa_i r_o = B_i e^{-\alpha_i r_o} \qquad (54a)$$

Furthermore, applying the condition of normalization, we get

$$1 = \int u_i^2(\vec{r}) d^3\vec{r} = \int_0^\infty F_i^2(r) dr = \int_0^{r_o} F_i^2 dr + \int_{r_o}^\infty F_i^2 dr$$
$$= \int_0^{r_o} A_i^2 \sin^2 \kappa_i r \, dr + \int_{r_o}^\infty B_i^2 e^{-2\alpha_i r} \, dr.$$

Integrating out gives

$$A_i^2 \frac{r_o}{2} - \frac{A_i^2}{4\kappa_i} \sin 2\kappa_i r_o + B_i^2 \frac{1}{2\alpha_i} e^{-2\alpha_i r_o} = 1 \qquad (54b)$$

From equations (54), we can determine A_i and B_i for a normalized spatial wave function $u_i(\vec{r})$.

For the spatial wave function of the final state, we must solve the general equation (47) with $V(r)$ given by eq.(49), i.e. we must solve the equation

$$\frac{d^2F_{f\ell}}{dr^2} + \left\{ \frac{M}{\hbar^2} \left[E_f + \frac{1}{2} (1+(-1)^\ell) V_o \right] - \frac{\ell(\ell+1)}{r^2} \right\} F_{f\ell} = 0 \qquad r < r_o \qquad (55a)$$

$$\frac{d^2F_{f\ell}}{dr^2} + \left\{ \frac{M}{\hbar^2} E_f - \frac{\ell(\ell+1)}{r^2} \right\} F_{f\ell} = 0 \qquad r > r_o \qquad (55b)$$

$F_{f\ell}$ must satisfy the same continuity conditions as F_i does. To satisfy the conditions at zero and infinity, we get the well-known solution

$$F_{f\ell} = A_{f\ell} \, r^{\frac{1}{2}} \, J_{\ell+\frac{1}{2}} (\lambda_{f\ell} r) \qquad r < r_o \qquad (56a)$$

$$F_{f\ell} = r^{\frac{1}{2}} \left[C_{f\ell} J_{\ell+\frac{1}{2}} (\alpha_f r) + D_{f\ell} J_{-\ell-\frac{1}{2}} (\alpha_f r) \right. \qquad r > r_o \qquad (56b)$$

in which

$$\lambda_{f\ell} = \sqrt{M \left[E_f + \frac{1}{2} (1+(-1)^\ell) \, V_o \right]} / \hbar \qquad (57a)$$

$$\alpha_f = \sqrt{ME_f} / \hbar \qquad (57b)$$

Applying the conditions of continuity and continuous derivative at $r = r_0$ gives

$$A_{f\ell}\, J_{\ell+\frac{1}{2}}(k_{f\ell} r_0) = C_{f\ell}\, J_{\ell+\frac{1}{2}}(\alpha_f r_0) + D_{f\ell}\, J_{-\ell-\frac{1}{2}}(\alpha_f r_0)\ . \qquad (58a)$$

$$A_{f\ell}\, k_{f\ell}\, J'_{\ell+\frac{1}{2}}(k_{f\ell} r_0) = C_{f\ell}\,\alpha_f\, J'_{\ell+\frac{1}{2}}(\alpha_f r_0) + D_{f\ell}\,\alpha_f\, J'_{-\ell-\frac{1}{2}}(\alpha_f r_0), \qquad (58b)$$

in which the prime means the differentiation of the Bessel function with respect to its argument.

To find the normalization condition, we make use of the following asymptotic forms of the Bessel functions as $r \to \infty$:

$$J_{\ell+\frac{1}{2}}(\alpha_f r) \longrightarrow \sqrt{\frac{2}{\pi\alpha_f r}}\,\sin\left(\alpha_f r - \frac{\pi}{2}\ell\right)$$

$$= (-1)^\ell \sqrt{\frac{2}{\pi\alpha_f r}}\,\sin\left(\alpha_f r + \frac{\pi}{2}\ell\right)$$

$$J_{-\ell-\frac{1}{2}}(\alpha_f r) \longrightarrow \sqrt{\frac{2}{\pi\alpha_f r}}\,\cos\left(\alpha_f r + \frac{\pi}{2}\ell\right)\ .$$

So for large r.

$$F_{f\ell} \longrightarrow r^{\frac{1}{2}} \sqrt{\frac{2}{\pi\alpha_f r}} \left[C_{f\ell}(-1)^\ell \sin\left(\alpha_f r + \frac{\pi}{2}\ell\right) + D_{f\ell}\cos\left(\alpha_f r + \frac{\pi}{2}\ell\right)\right]$$

Introducing δ such that

$$(-1)^\ell C_{f\ell} = \sqrt{C_{f\ell}^2 + D_{f\ell}^2}\,\sin\delta \qquad D_{f\ell} = \sqrt{C_{f\ell}^2 + D_{f\ell}^2}\,\cos\delta$$

gives $$F_{f\ell} \approx \sqrt{\frac{2}{\pi\alpha_f}}\sqrt{C_{f\ell}^2 + D_{f\ell}^2}\,\cos\left(\alpha_f r + \frac{\pi}{2}\ell - \delta\right)\ .$$

Since the final state is a state from the continuous spectrum, the wave function is not quadratically integrable. So we normalize it in a sphere of large radius $\mathcal{L}$. The condition of normalization is

$$1 = \int_0^{2\pi} d\varphi \int_0^{\pi} Y_{\ell m}^* Y_{\ell m} \sin\theta\, d\theta \int_0^{\mathcal{L}} F_{f\ell}^2(r)\,dr$$

$$\approx \frac{2}{\pi\alpha_f}\left(C_{f\ell}^2 + D_{f\ell}^2\right) \int_0^{\mathcal{L}} \cos^2\left(\alpha_f r + \frac{\pi}{2}\ell - \delta\right) dr\ .$$

Integrating out, we find

$$\frac{1}{\pi \alpha_f}\left(C_{f\ell}^{2}+D_{f\ell}^{2}\right)\mathcal{L}+\frac{1}{2\pi \alpha_f^{2}}\left(C_{f\ell}^{2}+D_{f\ell}^{2}\right)\sin 2\left(\alpha_f r+\frac{\pi}{2}\ell-\delta\right)\Big|_0^{\mathcal{L}}=1$$

Since we can choose $\mathcal{L}$ so large that the second term in the left hand side can be neglected in comparison with the first, we are then led to the following condition of normalization

$$C_{f\ell}^{2}+D_{f\ell}^{2}=\frac{\pi \alpha_f}{\mathcal{L}}. \tag{58d}$$

Eqs. (58) determine the three constants $A_{f\ell}$, $C_{f\ell}$ and $D_{f\ell}$ for a wave function normalized in a sphere of large radius $\mathcal{L}$. We must notice that this spherical volume is the same volume V in which we enclosed the photon field, as we have already stated in the text.

Now we have all the equations necessary for obtaining the initial and final spatial wave functions $u(\vec{r})$. We can then consider the matrix element of W given by eqs. (39 j,k,l) for the transition from the ground state of the deuteron to its disintegrated state. Remembering that the electric field of γ-rays is given by eq.(11) and the magnetic field is given by

$$\vec{H}=\sum_{k}\sum_{\rho=L,R}\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, i\left[\vec{k}\times\vec{e}_k^{\,\rho}\right]\left(a_{k\rho}\, e^{i\vec{k}\cdot\vec{x}}+a_k^{\dagger}\, e^{-i\vec{k}\cdot\vec{x}}\right), \tag{59}$$

we find from eqs. (39 j,k,l) the matrix element of W given by

$$\begin{aligned}\langle f|W|i\rangle=\langle f|W^E|i\rangle+\langle f|W^M|i\rangle=-\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\Big\{ i\omega\iint\Psi_f^{*}(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)\sum_{j=1}^{2}\Big[\frac{e}{c}\frac{1+\tau_{3j}}{2}(\vec{e}_k^{\,\rho}\cdot\vec{\xi}_j)\tfrac{1}{2}(e^{i\vec{k}\cdot\vec{x}_j}+e^{i\vec{k}\cdot\vec{R}})\Big]\Psi_i(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)d^3x_1d^3x_2\\ -i\iint\Psi_f^{*}(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)\sum_{j=1}^{2}\Big[\frac{e\hbar}{2Mc}\Big(\frac{1+\tau_{3j}}{2}\mu_p+\frac{1-\tau_{3j}}{2}\mu_N\Big)(\vec{\sigma}_j\cdot[\vec{k}\times\vec{e}_k^{\,\rho}])\tfrac{1}{3}(2e^{i\vec{k}\cdot\vec{x}_j}+e^{i\vec{k}\cdot\vec{R}})\Big]\Psi_i(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)d^3x_1d^3x_2\Big\}\end{aligned} \tag{60}$$

Here $$\langle f|W^E|i\rangle=-\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, i\omega\iint\Psi_f^{*}(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)\sum_{j=1}^{2}\Big[\frac{e}{c}\frac{1+\tau_{3j}}{2}(\vec{e}_k^{\,\rho}\cdot\vec{\xi}_j)\tfrac{1}{2}(e^{i\vec{k}\cdot\vec{x}_j}+e^{i\vec{k}\cdot\vec{R}})\Big]\Psi_i(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)d^3x_1d^3x_2 \tag{61a}$$

$$\langle f|W^M|i\rangle=-\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, i\iint\Psi_f^{*}(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)\sum_{j=1}^{2}\Big[\frac{e\hbar}{2Mc}\Big(\frac{1+\tau_{3j}}{2}\mu_p+\frac{1-\tau_{3j}}{2}\mu_N\Big)(\vec{\sigma}_j\cdot[\vec{k}\times\vec{e}_k^{\,\rho}])\tfrac{1}{3}(2e^{i\vec{k}\cdot\vec{x}_j}+e^{i\vec{k}\cdot\vec{R}})\Big]\Psi_i(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2)\,d^3x_1d^3x_2 \tag{61b}$$

where

$$\vec{\xi}_j=\vec{x}_j-\vec{R} \tag{61c}$$

Now we separate the wave functions of the nucleonic system into spatial, spin and isotopic spin parts, i.e.

$$\Psi_i(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2) = \varphi_i(\vec{x}_1,\vec{x}_2)\,\chi_s^{m_s}(\sigma_1,\sigma_2)\,\zeta_T^{m_T}(\tau_1,\tau_2) \qquad (62a)$$

$$\Psi_f(\vec{x}_1,\vec{x}_2;\sigma_1,\sigma_2;\tau_1,\tau_2) = \varphi_f(\vec{x}_1,\vec{x}_2)\,\chi^{m_s'}(\sigma_1,\sigma_2)\,\zeta_{T'}^{m_T'}(\tau_1,\tau_2) \qquad (62b)$$

Then we can see that the first integral does not vanish only when

$$\Delta m_s = m_s' - m_s = 0 \qquad (63a)$$

while the second does not vanish only when

$$\Delta m_s = m_s' - m_s = \pm 1 \qquad (63b)$$

if we assume that $\vec{k}$ is directed along z- axis. As eqs. (63a) and (63b) cannot be true simultaneously, we find that for every final state Ψ_f at least one of the operators W^E and W^M has a vanishing matrix element. We are thus led to the conclusion that

$$\left|\langle f|W|i\rangle\right|^2_{Av} = \left|\langle f|W^E|i\rangle+\langle f|W^M|i\rangle\right|^2_{Av} = \left|\langle f|W^E|i\rangle\right|^2_{Av}+\left|\langle f|W^M|i\rangle\right|^2_{Av}, \qquad (64)$$

where the meaning of the symbol Av has been clearly stated before. So these selection rules enable us to calculate the cross-sections for electric and magnetic effects separately, as pointed out in the text, since there are no cross terms present in the absolute square of $\langle f|W|i\rangle$.

From eq.(63) and the selection rules for W^E

$$\Delta S = 0 \qquad \Delta m_T = 0$$

it is seen that for electric transition Ψ_f must again be a triplet state as far as its σ- dependence concerned, while, like Ψ_i it must again be a state with

$$m_T' = 0$$

as far as its dependence on τ_1 and τ_2. Thence, by Pauli's principle, Ψ_f should be even in $\vec{x}_2 - \vec{x}_1$, and ℓ should be even, if $T' = 0$, while for similar reasons ℓ should be odd, if $T' = 1$. In the former case, the matrix elements $\langle \zeta_i^{*}, \tau_{2j} \zeta_0^{c} \rangle$ are easily seen to vanish. In the second case, $\langle \zeta_1^{0*}, \tau_{21} \zeta_0^{c} \rangle = -\langle \zeta_1^{0*}, \tau_{21} \zeta_0^{c} \rangle$, while in this case the matrix elements of isotopic unit matrices vanish. Thence, if we skip the factor $\langle \zeta_1^{0*}, \tau_{21} \zeta_0^{c} \rangle$ in W^E in the latter case (as its absolute squal is equal to 1 anyhow), we can reduce $\langle f|W^E|i\rangle$ to

$$\langle f|W^E|i\rangle = -\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\,(i\omega)\,\frac{e}{2c}\,\vec{e}^{\,\rho}_{k}\cdot\iint \varphi_f^{*}(\vec{x}_1,\vec{x}_2)\,e^{i\vec{k}\cdot\vec{R}}\left[\vec{\xi}_1\tfrac{1}{2}(e^{i\vec{k}\cdot\vec{\xi}}+1)\pm\vec{\xi}_2\tfrac{1}{2}(e^{i\vec{k}\cdot\vec{\xi}}+1)\right]\varphi_i(\vec{x}_1,\vec{x}_2)\,d^3\vec{x}_1\,d^3\vec{x}_2 \qquad (65a)$$

where the plus signs stands for $T' = 0$, $m'_{\varphi} = 0$, $\ell =$ even, and the minus sign for $T' = 1$, $m'_{\varphi} = 0$, $\ell =$ odd. For the magnetic matrix element, we find

$$\langle f|W^M|i\rangle = -\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\,\frac{e\hbar}{2Mc}\Big\{\Gamma^{P}_{s',m'_s,m_s;1}\,\Delta_{\varphi',m'_\varphi;1}\iint\varphi_f^{*}(\vec{x}_1,\vec{x}_2)\tfrac{1}{3}(2e^{i\vec{k}\cdot\vec{x}_1}+e^{i\vec{k}\cdot\vec{R}})\varphi_i(\vec{x}_1,\vec{x}_2)\,d^3\vec{x}_1\,d^3\vec{x}_2$$
$$+\Gamma^{P}_{s',m'_s,m_s;2}\,\Delta_{\varphi',m'_\varphi;2}\iint\varphi_f^{*}(\vec{x}_1,\vec{x}_2)\tfrac{1}{3}(2e^{i\vec{k}\cdot\vec{x}_2}+e^{i\vec{k}\cdot\vec{R}})\varphi_i(\vec{x}_1,\vec{x}_2)\,d^3\vec{x}_1\,d^3\vec{x}_2 \qquad (65b)$$

in which the Δ's are given by

$$\Delta_{\varphi',m'_\varphi;j} = \begin{cases} \frac{1}{2}(\mu_P-\mu_N) & \text{for } \varphi' = 1,\ m'_\varphi = 0 \\ \frac{1}{2}(\mu_P+\mu_N) & \text{for } \varphi' = 0,\ m'_\varphi = 0 \end{cases}\Big\}\ j = 1$$
$$\qquad\qquad \begin{cases} \frac{1}{2}(\mu_N-\mu_P) & \text{for } \varphi' = 1,\ m'_\varphi = 0 \\ \frac{1}{2}(\mu_N+\mu_P) & \text{for } \varphi' = 0,\ m'_\varphi = 0 \end{cases}\Big\}\ j = 2$$
$$\qquad\qquad 0 \quad \text{otherwise} \qquad (66a)$$

and the Γ's are tabulated as follows:

P = L								P = R							
j = 1				j = 2				j = 1				j = 2			
S', m'_s	$S\ m_s$ 1,1	$S\ m_s$ 1,0	$S\ m_s$ 1,-1	S', m'_s	$S\ m_s$ 1,1	$S\ m_s$ 1,0	$S\ m_s$ 1,-1	S', m'_s	$S\ m_s$ 1,1	$S\ m_s$ 1,0	$S\ m_s$ 1,-1	S', m'_s	$S\ m_s$ 1,1	$S\ m_s$ 1,0	$S\ m_s$ 1,-1
1, 1	0	-ik	0	1, 1	0	-ik	0	1, 1	0	0	0	1, 1	0	0	0
1, 0	0	0	-ik	1, 0	0	0	-ik	1, 0	ik	0	0	1, 0	ik	0	0
1,-1	0	0	0	1,-1	0	0	0	1,-1	0	ik	0	1,-1	0	ik	0
0, 0	0	0	-ik	0, 0	0	0	ik	0, 0	-ik	0	0	0, 0	ik	0	0

(66b)

Now we have to consider the integrals involved in $\langle f|W^E|i\rangle$ and $\langle f|W^M|i\rangle$. We first consider the integral involved in $\langle f|W^E|i\rangle$. From eqs. (39a) and (39b) and (61c), we have

$$\vec{r} = \vec{x}_2 - \vec{x}_1\ , \qquad \vec{R} = \frac{1}{2}\,(\vec{x}_1 + \vec{x}_2\ )$$

whence

$$\vec{x}_1 = \vec{R} - \frac{1}{2}\vec{r}\ , \qquad \vec{\xi}_1 = -\frac{1}{2}\vec{r} \tag{67a}$$

$$\vec{x}_2 = \vec{R} + \frac{1}{2}\vec{r}\ , \qquad \vec{\xi}_2 = +\frac{1}{2}\vec{r} \tag{67b}$$

As we have already done before, we separate the wave function $\varphi(\vec{x}_1, \vec{x}_2)$ into a product of function of $\vec{R}$ and a function of $\vec{r}$ i.e.

$$\varphi\ (\vec{x}_1, \vec{x}_2) = U\ (\vec{R})\ u(\vec{r}) \tag{68}$$

where $U(\vec{R})$ satisfies eq. (40a) and $u(\vec{r})$ satisfies eq. (45). After this substitution, eq. (65a) becomes

$$\langle f|W^E|i\rangle = i\frac{e}{2}\sqrt{\frac{2\pi\hbar\omega}{V}}\ \vec{\epsilon}^{\,\lambda}_{k}\cdot\iint U_f^*(\vec{R})\ u^*_{f\ell}(\vec{r})\ e^{i\vec{k}\cdot\vec{R}}\ \frac{\vec{r}}{4}\left\{ (e^{-i\vec{k}\cdot\frac{\vec{r}}{2}}+1) - (-1)^{\ell}(e^{i\vec{k}\cdot\frac{\vec{r}}{2}}+1)\right\} U_i(\vec{R})\ u_i(\vec{r})\,d^3\vec{R}\ d^3\vec{r} \tag{69}$$

Since $U_i(\vec{R}\ )$ and $U_f(\vec{R}\ )$ satisfy eq. (40a), they can, thus, be written as

$$U_i(\vec{R}\ ) = \frac{1}{\sqrt{V}}\ e^{\frac{i}{\hbar}\vec{P}_c\cdot\vec{R}} \qquad P_c = \sqrt{2M_cE_c} \tag{70a}$$

$$U_f(\vec{R}\ ) = \frac{1}{\sqrt{V}}\ e^{\frac{i}{\hbar}\vec{P}'_c\cdot\vec{R}} \qquad P'_c = \sqrt{2M_cE'_c} \tag{70b}$$

if we normalize them in the same volume V. Considering the law of conservation of momentum

$$\vec{P}'_c = \vec{P}_c + \hbar\vec{k}\ , \tag{71a}$$

we find

$$\int U_f^*(\vec{R}\ )\ e^{i\vec{k}\cdot\vec{R}}\ U_i(\vec{R}\ )\,d^3\vec{R} = 1\ , \tag{71b}$$

so that we get

$$\langle f|W^E|i\rangle = i\,\frac{e}{2}\sqrt{\frac{2\pi\hbar\omega}{V}}\left[+\int u^*_{f\ell}(\vec{r})\,\vec{\varepsilon}^{\,\rho}_k\cdot\frac{\vec{r}}{4}\,(e^{-i\vec{k}\cdot\frac{\vec{r}}{2}}+1)\,u_i(\vec{r})\,d^3\vec{r}\right.$$
$$\left.-(-1)^{\ell}\int u^*_{f\ell}(\vec{r})\,\vec{\varepsilon}^{\,\rho}_k\cdot\frac{\vec{r}}{4}\,(e^{i\vec{k}\cdot\frac{\vec{r}}{2}}+1)\,u_i(\vec{r})\,d^3\vec{r}\right] \quad (72)$$

Since the initial state is a ^3S state, we have

$$u_i(\vec{r}) = u_i(-\vec{r}) \quad (73a)$$

while

$$u_{f\ell}(r) = (-1)^{\ell}\,u_{f\ell}(-\vec{r})\ . \quad (73b)$$

Substituting eqs. (73a) and (73b) into the first term in eq.(72) and changing in this term the integration variable from $\vec{r}$ into $(-\vec{r})$, we see that the two integrals in eq.(72) are equal, so that we find simply

$$\langle f|W^E|i\rangle = (-1)^{\ell+1}\,i\,e\sqrt{\frac{2\pi\hbar\omega}{V}}\int u^*_{f\ell}(\vec{r})\,\vec{\varepsilon}^{\,\rho}_k\cdot\frac{\vec{r}}{4}\,(e^{i\vec{k}\cdot\frac{\vec{r}}{2}}+1)\,u_i(\vec{r})\,d^3\vec{r} \quad (74)$$

So the average square of $\langle f|W^E|i\rangle$ is

$$\left|\langle f|W^E|i\rangle\right|^2_{Av} = \frac{2\pi\hbar\omega}{V}e^2\left|\int u^*_{f\ell}(\vec{r})\,\vec{\varepsilon}^{\,\rho}_k\cdot\frac{\vec{r}}{4}\,(e^{i\vec{k}\cdot\frac{\vec{r}}{2}}+1)\,u_i(\vec{r})\,d^3\vec{r}\right|^2_{Av}, \quad (75)$$

which is eq.(14) in section 2.

Now we consider the integrals involved in $\langle f|W^M|i\rangle$. Proceeding in a similar way as what we did for the integral in $\langle f|W^E|i\rangle$ we can substitute eq.(68) with eqs. (70a) and (70b). We thus find

$$\langle f|W^M|i\rangle = -\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\,i\,\frac{e\hbar}{2Mc}\left\{\Gamma^{\rho}_{s',m_s',m_s;1}\,\Delta_{\tau',m_\tau';1}\int u^*_{f\ell}(\vec{r})\left(\frac{2e^{-i\vec{k}\cdot\frac{\vec{r}}{2}}+1}{3}\right)u_i(\vec{r})\,d^3\vec{r}\right.$$
$$\left.+\Gamma^{\rho}_{s',m_s',m_s;2}\,\Delta_{\tau',m_\tau';2}\int u^*_{f\ell}(\vec{r})\left(\frac{2e^{i\vec{k}\cdot\frac{\vec{r}}{2}}+1}{3}\right)u_i(\vec{r})\,d^3\vec{r}\right\},$$

or by eqs. (73a) and (73b),

$$\langle f|W^M|i\rangle = -\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\,i\,\frac{e\hbar}{2Mc}\,\lambda^{\rho}_{s',m_s',m_s;\tau',m_\tau';\ell}\int u^*_{f\ell}(\vec{r})\left(\frac{2e^{i\vec{k}\cdot\frac{\vec{r}}{2}}+1}{3}\right)u_i(\vec{r})\,d^3\vec{r} \quad (76)$$

where

$$\lambda^{\rho}_{s',m_s',m_s;\tau',m_\tau';\ell} = (-1)^{\ell}\,\Gamma^{\rho}_{s',m_s',m_s;1}\,\Delta_{\tau',m_\tau';1} + \Gamma^{\rho}_{s',m_s',m_s;2}\,\Delta_{\tau',m_\tau';2}\,. \quad (76a)$$

We have seen before that we cannot trust magnetic multipole transitions higher than the magnetic dipole transition; that is, use of $\vec{H}(0)$ is just as good as use of $\vec{H}(\vec{\xi})$. This means replacing $e^{i\vec{k}\cdot\vec{r}/2}$ by 1. In that case, the integral in eq.(76) simply vanishes by the orthogonality relation for $u_i(\vec{r})$ and $u_f(\vec{r})$. [From eq.(76), we find eq.(10) in section 2, if we take the average over the two directions of polarization and over the substates of spin of the initial state and ~~sum over the substates of spin of the initial state and~~ sum over the substates of spin and isotopic spin of the final state, and then use

$$|\lambda|^2_{Av} = k^2\left[\mu_P^2+\mu_N^2+\frac{2}{3}(-1)^{\ell}\mu_P\mu_N\right] \tag{77}$$

which follows from eq. (76a) by eqs. (66a) and (66b). Thus, we find, indeed, for $\ell = 0$

$$\left|\langle \ell=0|W^M|\ell=0\rangle\right|^2_{Av} = \frac{2\pi c^2\hbar}{V\omega}\left(\frac{e\hbar}{2Mc}\right)^2 k^2\left(\mu_P^2+\mu_N^2+\frac{2}{3}\mu_P\mu_N\right)\left|\int u^*_{f0}(\vec{r})\,u_i(\vec{r})\,d^3\vec{r}\right|^2,$$

which is eq.(10) in section 2.] If yet we want to calculate the magnetic quadrupole contribution, hoping the best for the eq.(9) for M, we find from eq.(76), by a Taylor's expansion of $e^{i\vec{k}\cdot\vec{r}/2}$:

$$\langle f|W^M|i\rangle = -\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, i\,\frac{e\hbar}{2Mc}\,\lambda^{P}_{S',m_S',m_S,T',m_T';\ell}\int u^*_{f\ell}(\vec{r})\left(1+\frac{1}{3}\,i\vec{k}\cdot\vec{r}\right)u_i(\vec{r})\,d^3\vec{r} \tag{77a}$$

For $\ell = 0$, we still find eq.(10) of section 2, this is, we find the vanishing of the magnetic dipole cross-section. For $\ell = 1$, we find

$$\langle \ell=1|W^M|\ell=0\rangle = \sqrt{\frac{2\pi c^2\hbar}{V\omega}}\,\frac{e\hbar}{2Mc}\,\frac{1}{3}\lambda^{P}_{S',m_S',m_S,T',m_T';\ell}\int u^*_{f\ell}(\vec{r})\;\vec{k}\cdot\vec{r}\;u_i(\vec{r})\,d^3\vec{r}. \tag{77b}$$

Substituting eq.(77) into the average square of eq.(77b), and performing the integration over the angles after the substitution of eq.(46), we find

$$\left|\langle \ell=1|W^M|\ell=0\rangle\right|^2_{Av} = \frac{1}{27}\,\frac{2\pi c^2\hbar}{V\omega}\left(\frac{e\hbar}{2Mc}\right)^2 k^4\left(\mu_P^2+\mu_N^2-\frac{2}{3}\mu_P\mu_N\right)\left|\int_0^\infty rF_{f1}(r)F_i(r)\,dr\right|^2 \tag{77c}$$

Thence, (compare eq.(84) below),

$$\sigma_{m\gamma} = \frac{V}{c\hbar^2} \left| W^M \right|^2_{Av} \sqrt{\frac{M}{E_f}}\, \mathcal{L} = \frac{\pi}{54} \frac{e^2}{\hbar c} \sqrt{\frac{Mc^2}{E_f}} \left(\frac{E_\gamma}{Mc^2}\right)^2 k \left(\mu_P^2 + \mu_N^2 - \frac{2}{3}\mu_P\mu_N\right) \cdot \mathcal{L} \left| \int_0^\infty r F_{f1}(r) F_i(r)\, dr \right|^2 \qquad (77d)$$

This is just the first member of eq.(29) in the section 2. Compare (77d) with eq.(23) we easily obtain the last member of eq.(29).

Now we are going to consider the matrix element for the electric quadrupole transition considered by Møller and Rosenfeld. They used for it the perturbation (see eq.(27))

$$-\frac{1}{2}\int \sum_{ij}\sum \rho \xi_i \xi_j (\nabla_j E_i)_0 \, d^3\vec{x} = \frac{1}{2}\int \sum_{ij}\sum e \Psi^\dagger \frac{1+\tau_z}{2} \Psi \xi_i \xi_j \frac{1}{c} \sum_k \sum_{\rho=1,2} \sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, \omega (e^\rho_k)_i k_j a_{k\rho} e^{i\vec{k}\cdot\vec{R}} d^3\vec{x}$$

$$+ \text{ complex conj.}$$

where Ψ is the quantized wave function of the nucleon. Then the matrix element of this for the electric quadrupole transition of the deuteron is

$$\frac{1}{2c}\int \sum_{ij}\sum e \Psi_f^* \sum_{n=1}^{2} \frac{1+\tau_{zn}}{2} \left[(x_i)_n - R_i\right]\left[(x_j)_n - R_j\right] \sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, \omega (\vec{e}^\rho_k)_i k_j e^{i\vec{k}\cdot\vec{R}} \Psi_i \, d^3\vec{x}_1\, d^3\vec{x}_2 \,.$$

Applying eqs. (67a) and (67b), we find, by $\tau_{z1}+\tau_{z2} = 0$,

$$-\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, \omega \frac{e}{2c}\int \sum_{ij}\sum \varphi_f^* \frac{1}{4} r_i r_j (\vec{e}^\rho_k)_i k_j \, e^{i\vec{k}\cdot\vec{R}} \varphi_i \, d^3\vec{R}\, d^3\vec{r}$$

$$= \sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, (i\omega) \frac{e}{2c}\int \varphi_f^* (\vec{e}^\rho_k \cdot \frac{\vec{r}}{2})(i\vec{k}\cdot\frac{\vec{r}}{2})\, e^{i\vec{k}\cdot\vec{R}} \varphi_i \, d^3\vec{R}\, d^3\vec{r}$$

After the integration over $\vec{R}$, and by the argument similar to that used for the transition (II), we find that the above expression reduces to

$$\sqrt{\frac{2\pi c^2\hbar}{V\omega}}\, (i\omega)\, \frac{e}{2c}\int u_{f2}^* \,(\vec{e}^\rho_k \cdot \frac{\vec{r}}{2})\, (i\vec{k}\cdot\frac{\vec{r}}{2})\, u_i \, d^3\vec{r}\,.$$

So the average square of this is given by

$$\frac{2\pi\hbar\omega e^2}{V}\left| \frac{1}{2}\int u_{f2}^* (\vec{r}) \,(\vec{e}^\rho_k \cdot \frac{\vec{r}}{2})\, (i\vec{k}\cdot\frac{\vec{r}}{2})\, u_i(\vec{r})\, d^3\vec{r} \right|^2_{Av} \qquad (78)$$

Eq.(78) is the contribution for the second term in a Taylor's expansion of the factor $e^{i\vec{k}\cdot\vec{r}}$ in the integrand appearing in the second member of eq.(25) as stated in section 2. (Note that the first term in this Taylor's expansion contributes nothing to the integral.) In fact, we used in the third member of eq.(25) not a Taylor's expansion, but the expansion (15). However, from the expansion (16b), we see that the term $\ell' = 1$ contributes the same as eq. (78), if we neglect terms of dimension of higher multipole contribution.

Finally, we shall derive eq.(21) in section 2, and the expression for the cross-section. The cross-section for the photo-disintegration of the deuteron can be easily found from the probability per sec. of the transition from the ground state of the deuteron to its disintegrated state, given by eq.(1), through the relation

$$\sigma = \frac{\text{probability of transition / sec.}}{\text{no. of incident photons /cm}^2\text{-sec.}} = \frac{w}{\text{incident flux}} \qquad (79)$$

Since the photon field is enclosed in a volume V, and there is only one incident photon, the number of incident photon per unit volume should be 1/V. The incident flux is then given by c/V, because the incident photon moves with velocity c. According to eq.(79) and eq.(1), the cross-section is given by

$$\sigma = \frac{V}{c} w = \frac{V}{c} \frac{2\pi}{\hbar} \left|W\right|^2_{Av} \rho\,(E) \qquad (80)$$

We have already found the expression for $\left|W\right|^2_{Av}$. Now we must find the expression for ρ (E) in order to get the complete expression of the cross-section.

The density ρ (E) of final states of the deuteron per unit change of energy can be easily obtained from the boundary conditions of the spatial wave function of the final state of the deuteron. As we have already mentioned before, the spatial wave function is normalized in a spherical volume $\bar{V}$ of very large radius $\mathcal{L}$, $F_{f\ell}(r)$ must satisfy the boundary conditions that

$$F_{f\ell}(r) = 0 \qquad \text{for } r = \mathcal{L} \tag{81}$$

From eqs.(81) and (58c), it follows that

$$\alpha_f \mathcal{L} + \frac{\pi}{2}\ell - \delta = (n + \frac{1}{2})\pi ,$$

or

$$\alpha_f \mathcal{L} = n\pi + \text{const.}$$

Thence by eq.(57b)

$$n = \frac{\alpha_f - \text{const.}}{\pi} = \mathcal{L}\frac{\sqrt{ME_f}}{\hbar\pi} + \text{const.} \tag{82}$$

The density of final states per unit change of energy is then

$$\rho(E) = \frac{dn}{dE_f} = \frac{1}{2\pi\hbar}\sqrt{\frac{M}{E_f}}\mathcal{L}, \tag{83}$$

which is the expression of eq.(21). With this expression for $\rho(E)$, the cross-section, given by eq.(80), can be written as

$$\sigma = \frac{V}{c}\frac{2\pi}{\hbar}|W|^2_{Av}\frac{1}{2\pi\hbar}\sqrt{\frac{M}{E_f}}\mathcal{L},$$

or

$$\sigma = \frac{V}{c\hbar^2}|W|^2_{Av}\sqrt{\frac{M}{E_f}}\mathcal{L}. \tag{84}$$

Substituting the different expressions of $|W|^2_{Av}$ for different transitions into eq.(84), we then get all the expressions for the cross-sections appearing in the text.

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VITA

Chia Hsien Teng, born May 19, 1924, Anhwei, China; graduated from National 9th High School in Szechuan, China in 1941; received B. S. in physics from National South-West Associated University in 1945; assistant at Peking University, Peking, China, from 1946-1948; graduate student and X - R fellow, Purdue University, 1948 up to the present.